How shall we talk about 'subject knowledge' for mathematics teaching?

Tim Rowland and Fay Turner

University of Cambridge

Sustained research into mathematics teacher knowledge over two decades, much of it in the UK, has drawn attention to the complexity of the knowledge base of mathematics teachers at all phases of education. Yet the official discourse of the topic remains rather blunt and simplistic. For example, remit 4 for the recent Williams review asked "What conceptual and subject knowledge of mathematics should be expected of primary school teachers ...". In this paper, we explore whether, and if so how, it might be possible to conceptualise and talk about knowledge for and in mathematics teaching, in ways that are acceptable to, and accessible by, education professionals and policy makers.

Keywords: mathematics; teacher knowledge; teacher education.

Introduction

The 'subject knowledge' of teachers of mathematics, especially but not only those in primary schools, has been a high-profile issue in the UK and beyond for more than a decade. At a local level, this has been evident in the programmes at BSRLM day conferences, including this one. Looking beyond the UK, a high proportion of papers recently published in the *Journal of Mathematics Teacher Education* have addressed mathematics teacher knowledge from some perspective, and a forthcoming special issue of *For the Learning of Mathematics* will be devoted to it. This new-found interest in teacher knowledge extends to policy makers and policy documents too, an exemplary case being the formulation of the English Standards for Initial Teacher Training from 1997 (DfEE 1997) to the present. Those commentators with more than the most superficial appreciation of the issue recognise that 'subject knowledge' for teaching has a pedagogical dimension (sometimes quaintly referred to as 'putting it across') as well as the more commonplace kind of knowledge acquired in studying mathematics at school or at university.

If, as is widely suggested, there is a 'problem' of some kind with teachers' mathematics subject knowledge, the nature of this knowledge has to be understood with a sophistication adequate to talk about it in conceptually useful ways, yet with sufficient simplicity to avoid being overwhelmed by it. A recent instance of lack of conceptual clarity in this respect can be seen in remit 4 for the recent Williams review, which asked "What conceptual and subject knowledge of mathematics should be expected of primary school teachers ...?". We acknowledge that this was penned by a politician, or a civil servant, but nevertheless we could ask why 'conceptual' and 'subject' are juxtaposed in this way. Is subject knowledge never conceptual?

Proposal for a minimal but adequate taxonomy

Shulman's (1986) classic taxonomy of teacher knowledge famously introduced the notion of pedagogical content knowledge (PCK) as an essential professional adjunct to subject-matter knowledge (SMK). The conceptual distinction between PCK and more generic pedagogic notions is that PCK is *specific to the subject-matter* being taught. Knowledge of commonplace mathematical misconceptions would be an example of PCK, in the case of the subject 'mathematics', and a demonstration that PCK goes beyond what the educated citizen would be expected to know. In a nutshell, SMK is "the amount and organization of the knowledge *per se* in the mind of the teacher" whereas PCK consists of "the ways of representing the subject which makes it comprehensible to others ....[it] also includes an understanding of what makes the learning of specific topics easy or difficult..." (Shulman 1986, 9).

Recent research by Deborah Ball and her colleagues at the University of Michigan unravels and clarifies these concepts. Shulman's SMK is separated into 'common content knowledge' (CCK) and 'specialized content knowledge' (SCK), while his 'pedagogical content knowledge' is divided into 'knowledge of content and students' and 'knowledge of content and teaching' (Ball, Thames and Phelps, submitted). In our view, the distinction between CCK and SCK within the category SMK is not very clear. A different distinction, due to Joseph Schwab (1978) but also explicit in Shulman's mental map (Shulman and Grossman 1988) is that between *substantive* and *syntactic* subject-matter knowledge. Substantive knowledge encompasses the key facts, concepts, principles, structures and explanatory frameworks in a discipline, whereas syntactic knowledge concerns the rules of evidence and warrants of truth within that discipline, the nature of enquiry in the field, and how new knowledge is introduced and accepted in that community – in short, *how to find out*. This distinction comes close to that between content (substantive) and process (syntactic) knowledge, although syntactic knowledge seems to entail greater epistemological awareness than process knowledge. Schwab's choice of the word 'syntactic' is perhaps unfortunate: 'syntax' suggests formal structure only, whereas the heuristics of enquiry are at the heart of the intended meaning. Nevertheless, we stick with the word. The substantive/syntactic distinction is important, not least because research with trainee primary teachers (Goulding et al 2002) suggests that syntactic knowledge cannot be adequately addressed or learned within initial teacher education.

Our proposal therefore is that a minimal-but-adequate conceptual taxonomy for policy and practice in mathematics teacher education is the three-fold framework comprising (*i*) substantive subject-matter knowledge (*ii*) syntactic subject-matter knowledge (*iii*) pedagogical content knowledge. These are exemplified in the following sections, with data from our own studies and from Alan Bishop's writing.

Substantive SMK: The case of Jason, and Elliot's Quarters

Jason was teaching elementary fraction concepts to a Year 3 (pupil age 7-8) class. The pupils each had a small oblong whiteboard and a dry-wipe pen. Jason asked them to "split" their individual whiteboards into two. Most of the children predictably drew a line through the centre of the oblong, parallel to one of the sides, but one boy, Elliott, drew a diagonal line. Jason praised him for his originality, and then asked the class to split their boards "into four". Again, most children drew two lines parallel to the sides, but Elliott drew the two diagonals. Jason's response was to bring Elliott's solution to the attention of the class, but to leave them to decide whether it is correct. He asks:

> Jason: What has Elliott done that is different to what Rebecca has done?
> Sophie: Because he's done the lines diagonally.
> Jason: Which one of these two has been split equally? […] Sam, has Elliott split his board into quarters?
> Sam: Um … yes … no …
> Jason: Your challenge for this lesson is to think about what Elliott's done, and think if Elliott has split this into equal quarters.

At that point, Jason returned the whiteboard to Elliott, and the question of whether it had been partitioned into quarters was not mentioned again. What makes this interesting mathematically is the fact that the four parts of Elliott's board are not congruent. The most immediate demands on Jason's SMK are therefore

i. by what criterion might non-congruent regions be deemed to be equal? The usual appeal to area seems very reasonable;

ii. *are* the areas of Elliott's four non-congruent triangles equal? How might Jason justify his answer to himself?

There then follow two further questions, each of which we would argue are SMK-related:

iii. How might Jason justify the equal-areas notion of equal non-congruent regions to these young children, if all the representations of unit fractions that they have previously experienced have been through partitions of shapes into congruent parts such as sectors of circles?

iv. How might Jason justify his answer to (ii) above to these 7-8 year-olds?

Our point is that questions such of these are, or could be, at the heart of Jason's response to Elliott, and the shaping of the direction of the lesson from that moment. The mathematics subject matter being taught is elementary, but the mathematical demands placed on Jason's *substantive* knowledge by Elliott's response are not at all elementary. The substantive knowledge in question (ii) entails comparing the areas of two triangles. That in (iv) is also partly substantive, but surely includes SMK too - having a sense of what 7-8 year-olds might be expected to know and understand. Jason's "challenge" to the class ("to think about what Elliott's done") could be motivated by a strategic decision to give the children some thinking time, but it might also be because he needs some time himself.

Syntactic SMK: Alan Bishop and the between-fraction

On more than one occasion, Alan Bishop (e.g. Bishop 2001, 244) has recounted an anecdote about a class of nine - and ten -year-olds who were asked to give a fraction between ½ and ¾. One girl answered $^2/_3$. When the teacher asked how she knows that it lies between the two given fractions she replied "Because 2 is between the 1 and the 3, and on the bottom the 3 lies between the 2 and the 4". Bishop uses the anecdote to illustrate how a teacher's response is determined by the values that they espouse. From our immediate perspective, it is of interest because

i. a claim is implicit in the girl's answer: that if $^a/_e < {}^c/_g$, $a < b < c$ and $e < f < g$, then $^a/_e < {}^b/_f < {}^c/_g$

ii. it would not be surprising if the teacher does not know whether the claim is true (for all values of the positive integer variables).

In connection with (ii), the claim hints at the notion of the Farey Mean of two fractions, (sometimes called their 'mediant'), but this topic is not standard material for the school curriculum, in England at least. More to the point, it would be possible to have gained a first degree, or even a doctorate, in mathematics without having encountered it. The teacher is therefore driven to call upon mathematical knowledge resources concerned with knowing-how (to investigate the claim) as opposed to knowing-that (the claim is true or not). These kinds of SMK resources are *syntactic* knowledge.

Pedagogical Content Knowledge: Amy teaches counting

Most adults have no difficulty with counting, therefore it might be supposed that the knowledge needed for teaching this aspect of primary mathematics is commonplace and not specific to teaching. There is, however, a body of knowledge concerning the nature and the pedagogy of counting the educated citizen does not need, and could not be expected to know. An important part of this knowledge was identified by Gellman and Gellistel (1978), who proposed five principles that children must be able to demonstrate in order to be able to count meaningfully. The first three of Gelman and Gellistel's principles relate to 'how to count' and the other two to 'what can be counted'.

The 'how to count' principles are:

- The *one-one principle* involves giving each item in a set a different counting word. This is not enough of itself however as a child might count three items as 1, 3, 4 one time and as 1, 2, 4 another.
- The *stable order principle* involves knowing that the order of the counting words must always be the same. A child may still not count 'correctly' if for instance they consistently count three objects as 1, 2, 4. This is a case of convention rather than understanding, however since it is important to be able to communicate meaning to others, the child needs to learn the conventional order of number names rather than using their own.
- The *cardinal principle* refers to the idea that the last number in a count is the answer to the question 'how many?' After counting a set of objects, when asked 'how many?' a child might begin the count again. This is because they see the answer to the question 'how many?' as the process of counting rather than the number they reach at the end.

The 'what can be counted principles' are:

- The *abstraction principle* involves children recognising that any entities may be counted whether they are physical or abstract and that these entities do not have to be similar to be included in a count. This recognition demonstrates that the children are able to apply the abstract idea of number to an object in the same way that they might apply a physical property such as colour.
- The *order-irrelevance principle* involves knowing that it does not matter in which order items are counted. Children who demonstrate this principle are able to start and finish at any item or make any specified item a given number in the count. They do not see the number names as labels specific to items but as a means to reaching the answer to 'how many?'

In our observations of the teaching of counting it has been apparent that having the kind of knowledge about counting discussed above does influence practice. One teacher, Amy, made it clear that she had started from her knowledge of Gelman and Galistel's principles for counting when planning her lesson, and this was apparent in her teaching. Her lesson began with 'rhythm counting', saying the number names consistently in the conventional order. This was helping the children to understand the *stable order principle.* Amy praised the children for touching or pointing to each object just once, helping to reinforce the *one-to-one principle.* She encouraged the children to put the objects in a line or to move them to a new pile as they counted, both strategies for ensuring adherence to the *one-to-one principle*. In order to reinforce the *cardinal principle*, Amy asked the children to say how many there were after they had finished their count. She also made this principle explicit by telling the children that the last number they had said was the answer to the question 'How many?'

When they had completed counting the objects in the different boxes, Amy asked questions about which held more and which held less. She also asked them to put the boxes in line from the smallest to the largest number of objects. To do this the children had to ignore the fact that the objects in each box were different and also to ignore the size and colour of the boxes. This related to Gelman and Gallistel's final two principles of *abstraction* and *irrelevance.* The children had to understand the idea that the number of the count was an abstract property of the set of objects in the boxes. They needed to understand that the type of objects and that the colour and size of the boxes were irrelevant.

Though the topic of this lesson appeared to be a very simple one, Amy drew on her bank of specialist theoretical knowledge about what children needed to know and needed to be able to do in order to count meaningfully. In a post-lesson discussion she said that she had planned her lesson around the principles for counting and was able to name the first three, though she did not remember these had been proposed by Gelman and Gellistel. During a group discussion later, with other beginning teachers, Amy claimed that when planning her teaching she thought about an essay on counting she had written during her training year which would have made reference to these principles. This is an example of teaching in which we saw a teacher's *pedagogical content knowledge* having a positive impact on their teaching and the children's learning. It illustrates the importance of PCK in an aspect of mathematics where non-teachers would be expected to have adequate substantive content knowledge.

Conclusion

We suggest that the three-fold framework comprising (*i*) substantive subject-matter knowledge (*ii*) syntactic subject-matter knowledge (*iii*) pedagogical content knowledge, though minimal, is adequate for discussion of mathematical knowledge in teaching. We used data from primary classrooms to illustrate each of the three aspects of the framework. We suggest that it can be applied more generally to the analysis of mathematics teaching, and also that it is sufficiently focused to identify how very different kinds of teacher knowledge are made visible in teaching. We would be interested to know, in the first instance, whether other teacher educators share our view concerning the adequacy of this framework, and whether they find it useful – as we do - in their own work in mathematics teacher education.

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