The Development of a Semantic Model for learning Mathematics

Michael Peters

University of Plymouth/Cornwall College

The semantic model described in this paper is based on ones developed for arithmetic (e.g. McCloskey et al 1985, Cohene and Dehaene 1995), natural language processing (Fodor 1975, Chomsky 1981) and work by the author on how learners parse mathematical structures. The semantic model highlights the importance of the parsing process and the relationship between this process and the mathematical lexicon/grammar. It concludes by demonstrating that for a learner to become an efficient, competent mathematician a process of top-down parsing is essential.

Keywords: Parsing, Lexicon, Grammar, Cognitive Pathway

Introduction

In order to make sense of our environment we use language to communicate our thoughts, feelings and ideas. Words are used to convey the idea that thoughts contain concepts and concepts are about categories. The terms concept and category are used in the sense defined by Braisby (2005); concept refers to something in the mind and category to those things in the external world which the concept refers too. Within a particular community a socially mediated set of rules (grammar) and definitions (lexicon) evolve over time. In order for a person to become a member of such a community the grammar and the lexicon have to be learnt. A person born into the community naturally acquires the lexicon and grammar as they mature, going through a process of refinement until they reach a point where they are deemed to be competent. Mathematical language comprises of natural language, technical language and symbols. In order for learners to learn mathematics they must acquire the lexicon and grammars associated with the different operations and procedures associated with mathematics. Language acquisition raises such questions as: how is language acquired? Do humans have a predisposition for language acquisition? These topics have been well researched (e.g. Chomsky (2006), Pinker (1994), Hirsh-Pasek and Golinkoff (1996)) in terms of natural language but not so much in terms of mathematical language and the learning of mathematics.

The arithmetic and natural language processing models of McCloskey et al (1985), Cohen and Dehaene (1995), Fodor (1975) and Chomsky (1981)were developed either from observation and analysis or by using data gathered from patients with some form of cognitive disorder. The model developed in this paper, although based on these models, was further developed from data gathered from the observation of learners, semi-structured interviews and analysis of learners' answers to the Chelsea Diagnostic Test (CDT) for Algebra.

Research Methodology

A case study approach was adopted for this research. The rationale for this approach include the size of the learner groups (fifteen learners) participating in the research and the techniques selected for data collection. Essentially three formal methods were used to gather data: the Chelsea Diagnostic Test for Algebra (CDT), videoed semi-structured interviews and observations of the learners. The observations also included the video recording of the learners' computer monitors when they were engaged with using a mathematics computer program (Derive 6). The CDT results were tabulated and categorised according to the four levels used by the original research team (Küchemann 1981). The levels were only used as a reference to indicate the mathematical maturity of the learner and provide a measure of the change in the learners' knowledge and skills. Of more interest were the formulations of their answers. These gave an insight into how they transcoded the questions, interpreted and subsequently answered them. The data from the interviews and observations provided the necessary context in which the learners' answers could be interpreted.

Literature Review

Analysis of the English language reveals how complicated the process of communication is. In order to try and understand language as a communicative medium it is necessary to introduce some terminology which in itself is the subject of debate. The terms concept and category are used in the following discussion; concept refers to something in the mind and category to those things in the external world which the concept is about (Braisby 2005).

To make sense of language, concepts and categories are used to interpret and to understand phenomena. Words are used to convey the idea that thoughts contain concepts, and the concepts are about categories. The use of concepts to classify (semantic classification) can be viewed as a further kind of recognition. Concepts can also be viewed as the basic units of semantic memory, where long term facts are stored. Braisby (2005) used the idea of a 'mental lexicon' to describe where 'lexical concepts' (defined as single word concepts) are stored. It is thought that the process of understanding language involves retrieving lexical concepts and selecting the appropriate one. As Braisby (2005) points out, if someone was asked to describe a cat they would most probably infer properties from the class of animals (warm blooded, has eyes, legs etc). This aspect of using concepts reduces the cognitive load, since it is not necessary to remember all the inherited properties a class has from a super class.

It can be argued that the meaning of a sentence is derived from the meanings of its constituent words and the way in which they are combined (Davies 2006). This notion of compositionality enables us to make sense of novel sentences provided they are composed of familiar words. In the case of natural language, meaning can also be derived from the overall context of the sentence and/or from the environment in which it is used (see Anderson (1997) on use of language in the workplace). The meanings attached to the constituent words of a sentence rely upon a process of recognition. Understanding a sentence is much more difficult and relies upon a process known as parsing.

Parsing

Parsing is the process of taking a word and deducing its grammatical or syntactic role in the current sentence. This can be stated in a more formal way by saying that parsing a string (sentence) means to reconstruct the syntactic tree(s) that indicates how the string can be produced from the given grammar. This process does not recover the meaning of the sentence but facilitates a thematic role assignment for the sentence i.e. identifying the subject, verb and object (Gaskell 2005).

In his discussion of language processing, Gaskell (2005) asks the question: does human parsing involve the process of identifying major syntactic boundaries and then assign a syntactic structure or does it use a process of incremental parsing? This really is a question of whether top-down parsing, where a word sequence is hypothesized to be a sentence which conforms to a phrase structure or bottom-up parsing, where words are grouped into phrases until a plausible sentence is possibly achieved. Evidence suggests (Cooper 2002) that both top-down and bottom-up parsing are employed at different stages of human sentence processing. In a study by Tyler and Marslen-Wilson (1977), designed to explore the role of context in parsing, they demonstrated by using ambiguous phrases such as 'landing planes' that context did make a difference. For example, when the phrase was preceded by 'If you walk too near the runway,…' 'landing' was interpreted as an adjective (e.g. landing planes are dangerous) whereas if it was preceded by 'if you are trained as a pilot,…' the interpretation was more likely to be a verb (e.g. landing planes is easy). Their conclusion was that parsing was an incremental process and did not rely upon reaching syntactic boundaries, since the preceding context did have an influence on word expectation, whereas if parsing relied upon the identification of syntactic boundaries word expectation would not be an issue. This incremental processing model seemed to demonstrate that a sentence is constructed from a process of refining the set of plausible structures which in turn has an effect on word expectation.

Frazier (1979) proposed a model of parsing that become known as the garden path model of parsing. This model assumed an incremental parser which implies that as each word is perceived, a syntactic role is assigned to it. Due to the flexibility in the syntactic role of English words this can lead to the situation where the parser makes a wrong choice as to the role of the word in the sentence. In these ambiguous situations the listener is led 'down the garden path' and, unless as the sentence proceeds the listener derives a sensible interpretation, a misconception is achieved. The garden path model assumes the parser works in a serial fashion and therefore only one potential parse of a sentence is maintained based on the syntactic role of autonomous words. This model also implies if an incorrect parse is obtained the whole sentence needs to be re-parsed. Other models of human parsing have focused on the parallel and interactive nature of the activity and are commonly referred to as constraint-based models (e.g MacDonald and Pearlmutter 1994). They suggest that more than one potential parse of a sentence is evaluated simultaneously and other factors are involved in the parsing process. Semantic plausibility, lexical frequency, prosody are examples of factors which could influence the human parsing process.

Reading and Parsing Mathematics

People often employ strategies with natural language to enable them to understand something without necessarily interpreting every symbol. Mathematics differs considerably in this aspect to language. It is often necessary for the reader especially when learning a new 'piece' of mathematics to read every sign, consider its syntactic meaning, search their mental lexicon for a semantic meaning, and then relate the symbols to provide a meaningful result. This in itself can be problematic; many signs within mathematics have multiple meanings. For example, the equals sign is often used to indicate assignment, the result of a process or equivalence. Within mathematics, the context and hence the interpretation of the symbols is not always as straight forward. In many instances this does not create problems for the learner when they are using a particular aspect of mathematics but, due to the hierarchical nature of mathematics, when the learner progresses onto a more advanced level this 'fuzzy' understanding of a basic concept or process can create major problems. Making sense within mathematics is a much more complicated and involved structure. This process of reducing descriptions of operations etc to symbols enables mathematics to be a very powerful tool in understanding our world, but at the cost of making the learning of mathematics harder.

Model Adopted and Rationale

The model which this research uses is based on process oriented theories. The adopted model, since it is dealing with adult learners acquiring a mathematical grammar, acknowledges the influence of the prevalent mathematical culture found within institutions. This culture includes the notion of exploring mathematics and learners discovering for themselves mathematical 'truths'. The rationale for adopting this approach is based on the findings of Hirsh-Pasek and Golinkoff (1996). Their experimental approach to investigating the idea of children having an inbuilt sensitivity to acquiring language demonstrates a uniquely human attribute. They found that children, in order to acquire a grammar, must know how to package words together to formulate a meaning from a sequence of words. They showed children were able to identify the relationships between word units in order to derive an interpretation based on semantic plausibility.

Semantic Model

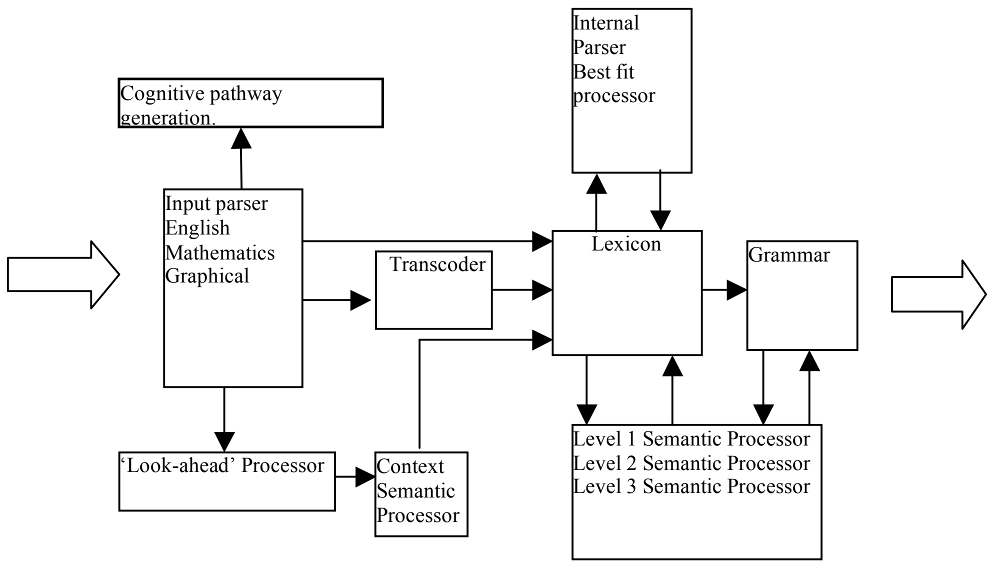

Figure 1,The Semantic Model

The model developed demonstrates a relationship between the parser, lexicon and grammars and suggests the most efficient cognitive pathway for a particular learner is initiated on the first parse of a mathematical statement. It is possible that these pathways are initially established when the learner first encounters a new mathematical construction. Depending upon the source of this new information, the lexical/grammatical entry is given a 'trustworthy' rating i.e. the more the source is trusted the higher the weighting. These weightings are subsequently amended when the entries are confirmed to be correct or decreased if the learner encounters a situation where the lexicon or the grammar entry is conceived to be 'not fit for purpose'. In some instances where the learner cannot find an entry that precisely matches the parsed entry a point of ambiguity is reached which must be resolved. The resolution could be finding an entry that is a 'best fit' which is possibly a 'folk' definition and not the correct mathematical one.

Mathematical expressions make use of a plethora of signs and implications. For example, in a simple addition problem the absence of a sign in front of a numeral implies a positive value. The learner has to initiate a 'look ahead' process to determine the purpose of the sign and, with reference to its context, form a symbol. This information is used in conjunction with the lexicon and grammar and is used to initiate the cognitive pathway and hence form a resolution.

The model depicts that semantic processing is performed at different levels, level one to level three, in reality this is a dynamic process. In the early stages of learning a mathematical topic the learner has to semantically process every detail of the construction. This is the stage where the learner is still working in a bottom-up parsing mode. Once the learner becomes competent and hence confident a top-down approach to parsing is used with a consequential decrease in semantic processing. The problem which required level three semantic processing becomes a level one with the concomitant decrease in cognitive load.

Summary

The learning of mathematical language is similar to the acquisition of natural language. Major differences such as lack of 'practice time', no inherent 'sympathy' for mathematical language acquisition exist but the process of parsing follows similar rules. The child learning their natural language initially adopts a bottom-up approach to natural language acquisition i.e. each word is considered in turn. The route to mathematical language competence is similar. In the initial stages of learning mathematics, learners tend to focus on the atomic structures and have to explicitly access the lexicon to discover the meaning of each part of a construction. This bottom-up approach is cognitively demanding and, as with language learning, the learner fails to grasp the overall meaning of the 'mathematical sentence'. Each stage of the parse requires mental effort to recall and retrieve the lexical entry and to find the associated grammar. It also requires an implicit feedback process, especially if the learner is not confident, whereby the concepts and processes are checked to see if they are the correct ones. In the case of multiple lexical entries for a particular construct the learner has to select one from a list, form a link to the appropriate grammar and then rely upon experience (i.e. constructions stored in memory) to determine the 'goodness-of-fit'. There is an inherent problem with the progression from bottom-up to top-down parsing. Although it is a far more efficient mental process it relies upon prediction and hence expectation. The prediction and expectation elements of top-down parsing rely upon the correct cognitive pathways being in place. If this is not case, the learner will produce erroneous answers and will not be able to analyse their answer to see why it is wrong. The learner, in order to progress, would have to return to a bottom-up parsing approach and 'reroute' the cognitive pathway.

Conclusions

In order for a learner of mathematics to appreciate the overall sense of a mathematical problem a top-down parsing process must be used. If the learner remains at the stage of bottom-up parsing and focuses on the atomic structures, a considerable amount of cognitive resources are employed and the learner fails to see the overall significance of the mathematical expression.

A common comment made by learners observed by the author was: '*the examples were easy, but the questions are a lot harder'*. The writer of the problem, normally a competent mathematician and/or teacher, has developed sufficiently to employ top-down parsing strategies and is therefore able to 'see' the relationship between the simple example and the harder problem designed to 'stretch' the learner. To facilitate top-down parsing the learner has to combine individual mathematical concepts to form 'super' concepts and consequently be able to predict the content of expressions. Once this stage is reached the learner as reached a level of competency and is at the stage where more complex constructions are capable of being dealt with. It is often the case where the learner seems to have reached this point in their mathematical development but, in reality they are still moving towards a top-down parsing strategy and have not quite reached the level of competency required to apply it reliably.

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