What are the argumentative activities associated with proof?

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Our goal in this paper is to identify the different argumentative activities associated with the notion of mathematical proof. Having identified the different activities we present the results of a bibliographic study designed to explore the extent to which each of these activities has been researched in the field of mathematics education. We conclude by arguing that the comprehension and presentation of given arguments are important, but under-researched mathematical activities related to proof.

Keywords*:* Argumentation; Bibliographic study; Proof.

Introduction

Proof is widely agreed to be central to the activity of mathematicians, however it is also a notoriously difficult concept for students to learn (e.g. Mariotti 2006). These two factors have led to recurring discussions of proof and proving in the mathematics education literature. One of the most influential frameworks used to situate such discussions is the theory of *proof schemes* (Harel and Sowder 1998). Harel and Sowder defined a person's proof scheme to be the processes they use to become certain of the truth of a mathematical statement, and to convince others of this certain truth. In an exhaustive study they provided a detailed classification of the different proof schemes used by college students, noting that many used non-deductive schemes.

Of particular interest for our purposes is that Harel and Sowder considered a wide variety of situations when studying their students proof schemes; including problem exploration activities, 'proof that…' tasks, 'true or false' tasks and 'explain why…' tasks. It is unclear whether, in each of these situations, students focus solely on the truth of statements (or, if they do, whether they focus solely on gaining certainty). Indeed, Healy and Hoyles (2000) noticed that students would often prefer different arguments for presentation to a teacher than they would for convincing themselves, suggesting that the task context is an important factor when analysing proving activities.

Our primary goal in this paper is to classify different activities associated with proof, with reference to task contexts. Our underlying assumption is that each of these different activities could, in principle, cause different behaviour (whether this is the case or not, of course, is an empirical matter).

Activities concerning proof

When laying out a preliminary map of the different activities which mathematicians engage in, Giaquinto (2005) suggested that for any piece of mathematics there are three associated general activities: making it, presenting it, and taking it in. Within the

context of proof and argumentation these three general activities correspond to: constructing a novel argument, presenting an available argument, and reading a given argument. However, the behaviour associated with these three distinct activities is likely to vary between contexts. A mathematician presenting an argument as part of a journal article, for example, may well behave differently from if she were presenting an argument in an undergraduate lecture course.

One of the reasons that different contexts produce different behaviour is that the goal the individual has in mind at the time is likely to vary. De Villiers (1990), following Bell (1976), proposed that proof has five main functions or goals:

- Verification: establishing the truth of a statement;
- Explanation: providing insight into why a statement is true;
- Systematisation: organising results into a deductive system;
- Discovery: the discovery or invention of new results;
- Communication: the transmission of mathematical knowledge.

De Villiers's categorisation suggests that each of the three general activities related to mathematical argumentation can be performed with these different functions in mind. For example, someone may present an argument in order to persuade a given audience of the conclusion's truth, to provide them with insight into why it is true, or to demonstrate the argument's validity in a given system. Similarly, one might read an argument with the intention of understanding it, or in order to evaluate how persuasive, explanatory, or valid it is.

Our framework of sub-activities concerning proof emerges from considering how De Villiers's goals may guide each of Giaquito's general types of activities. This leads us to differentiate between three distinct types of construction activities, which we call *exploration of a problem, estimation of truth of a conjecture,* and the *justification of a statement estimated to be true.* Similarly, we identify two main reading activities: the *comprehension of a given argument* and the *evaluation of an argument* with respect to a given set of criteria. Finally, we differentiate between presentations in which a given argument is used to: *convince a given audience* of the argument's claim*, explain to a given audience* why the claim is true*, demonstrate the argument's validity to a given audience* and *demonstrate to an expert one's understanding* of the given argument. Table 1 summarises this framework in terms of what is the initial given situation in each sub-activity, its particular goal and its expected product.

Construction	Problem exploration	Estimation of truth	Justification
Given	A problem situation	A conjecture	A statement estimated to be true
Goal	Answer an open-ended question	Estimate the truth of the conjecture (e.g. true or false questions)	Give reasons justifying the statement (e.g. "prove that" questions)
Product	An argument with a new statement as claim	An argument with the conjecture as claim and a non-neutral qualifier.	An argument with the given statement as claim.

Table 1a. Activities associated with the construction of a novel argument.

Reading	Comprehension	Evaluation
Given	An argument	An argument and a set of criteria
Goal	Understand the argument	Assess the argument against the given criteria (e.g. 'is it convincing?', 'is it a proof?' questions)
Product	Possibly sub-arguments with the given argument's statements as claims.	An assessment (yes/no or continuous) and possibly a justification of the assessment.

Table 1b. Activities associated with the reading of a given (external) argument.

Presentation	Conviction of an audience	Explanation to an audience	Demonstration of validity	Demonstration of understanding
Given	An argument and an audience			An argument and an expert
Goal	Convince the audience that the claim is true using the argument	Explain to the audience why the statement is true using the argument	Demonstrate to the audience the validity of the argument	Demonstrate to the expert that one understands the argument
Product	A variation of the given argument			

Table 1c. Activities associated with the presentation of a given (external) argument.

Activities discussed in mathematics education: a bibliographical study

In order to explore the extent to which each of these activities has been researched in the field of mathematics education, we conducted a bibliographical study aimed to analyse the type of tasks discussed and employed in empirical research in the field.

Methods

The bibliographical study involved conducting a search of education articles discussing the notion of mathematical proof, filtering those that discussed tasks related to this notion (either to illustrate a theoretical viewpoint or as part of an instrument in an empirical study), and classifying these tasks according to their given conditions, their implicit goal and expected product.

In order to avoid sampling biases, we conducted the search using only one database, the Education Resources Information Center (ERIC), a large online digital library of education research. We searched the ERIC collection for all journal articles which contained the keywords 'proof' and 'mathematics' (i.e. Publication Date: pre-1966 to 2008; Keywords (all fields): *proof* AND *mathematics*; Publication Type: *Journal Articles*; Education Level: *Any Education Level*).

This search produced a list of 610 articles, which included a large number of articles that were irrelevant to our study (e.g. *American Mathematical Monthly* articles presenting actual mathematical proofs, and non-empirical articles published in professional journals as *Mathematical Teacher*). Therefore, from these 610 articles,

we selected those that appeared in seven journals chosen for their tradition of publishing empirical mathematics education research: *Cognition & Instruction*, *Educational Studies in Mathematics, For the Learning of Mathematics, Journal of Mathematical Behaviour, Journal for Research in Mathematics Education, Mathematical Thinking and Learning* and *ZDM.* Our final sample contained the 102 articles in the original list that that been published in one of these seven journals.

We then searched each of these 102 articles for any task related to the notion of proof. Each of these tasks was classified into one or more of the nine sub-activities in our framework, depending on the given conditions of the task, its main goal and product. For instance, in one of the articles in our sample, Recio and Godino (2001) discussed undergraduate students' responses to the following task: "Prove that the difference between the squares of every two consecutive natural numbers is always an odd number, and that it is equal to the sum of these numbers." This particular task gave students a specific statement and asked them to prove it. Therefore, this task was classified as involving the justification of a given statement. Selden and Selden (2003) asked mathematics undergraduates to read purported proofs and decide whether or not they were proofs. Clearly, Selden and Selden's task involved reading a given argument, with the goal of evaluating it against the criteria of validity. Therefore, these tasks were classified as involving the validation of a given argument.

Results

Table 2 presents the number of articles that discussed at least one task related to a sub-activity in our framework. From those articles in our sample that discussed specific tasks, the majority (55) addressed students' construction of novel arguments, some (21) involved students' reading of given arguments and none discussed the presentation of a given (external) argument. In particular, only 3 articles addressed tasks related to the comprehension of a given argument and none of the articles discussed tasks related to the presentation of an argument to demonstrate students' understanding of it.

Main activity	**Sub-activity**		**N**
Construction	Exploration of a problem		24
	Estimation of the truth of a conjecture		11
	Justification of a statement		20
Reading	Comprehension		3
	Evaluation	Miscellaneous (e.g. 'is it convincing, explanatory?')	9
		Validation (e.g. 'is it a proof?')	9
Presentation	Conviction of an audience		0
	Explanation to an audience		0
	Demonstration of validity		0
	Demonstration of understanding		0

Table 2. Number of articles discussing tasks in each sub-activity.

Discussion

What argumentative activities related to the notion of proof do students normally engage in when learning mathematics? This is an interesting question that could be studied empirically. Nevertheless, from our own experiences as mathematics students and teachers at the school and undergraduate level we suspect that there are three main proving activities in which students regularly engage when learning mathematics:

- Construction of novel arguments: in the exploration of a given problem situation, while estimating the truth of a given conjecture (e.g. when addressing 'true or false' questions), or when asked to justify/prove a statement they had not seen before (mainly in classroom activities or assignments);
- Reading arguments given by teachers/lecturers or presented in books with the objective of understanding them;
- Presenting arguments that they had previously read in order to explain these arguments to their peers (during class), or to demonstrate to their teachers that they understand them (normally in exams).

It is hard to determine the relative importance of each of these activities in the learning of mathematics without data from a detailed empirical study on the types of argumentative activities that students engage in during classroom activities, when working on homework assignments, and taking exams. However, we hypothesise that (i) the comprehension of given mathematical arguments and (ii) the presentation of these arguments to demonstrate one's understanding of them, are two of the key activities involved in the assessment of undergraduate students' proving skills: students spend long periods of time trying to understand/memorise proofs in mathematics textbooks and lecture notes, and then present these arguments (or parts of them) to their teachers in exams with the aim of demonstrating their understanding.

If this is indeed the case, our findings suggest that we, mathematics educators, know very little about students' behaviour in some of the main types of activities involved in the assessment of their proving skills, which in turn may become the type of activities many students focus on, precisely because of their involvement in assessment.

Conclusions

We have presented a framework of activities associated to the notion of proof, which builds on the specific given conditions and the goals guiding the construction of a novel argument, the reading of an argument and the presentation of a given argument to a given audience.

We have also discussed the findings of a bibliographical study on the type of tasks discussed (and employed) in empirical research in mathematics education. These findings suggest that researchers in the field have tended to concentrate on understanding a relatively small subset of the activities associated with mathematical argumentation and proof. In particular, we have suggested that two key argumentative activities involved in the assessment of students' proving skills have yet to receive substantial research attention: the comprehension of given mathematical arguments and the presentation of an external argument to demonstrate one's understanding of it.

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