Imagery & Awareness

John Mason

Open University & University of Oxford

I am interested in the core or key awarenesses which underpin the various topics, techniques and concepts of school mathematics. I take awareness to be the basis for action, which means it need not be conscious. Every action undertaken in mathematics, whether rotating a figure, reducing a fraction, counting on, or reasoning about a trigonometric identity is informed by and made possible because of previous experience with similar actions in the past, from which grows the awareness. I am also interested in how attention moves, and what its movement reveals about awareness. In this session I offered people some tasks through which to experience some movements of their attention, and so perhaps to enable awareness of their awarenesses to come to articulation.

Introduction

I use a phenomenological or experiential approach, so participants were invited to engage in tasks and to reflect on that experience. The data I offered as part of my reporting of the results of my enquiries, was what they noticed happening for them when engaging in the tasks I offered. My approach involves the ongoing refinement of tasks which provoke experiences, and the ongoing checking of whether what I have noticed is noticeable to others (Mason 2002).

Task 1

> What shape would you use with learners so that they could experience the way that the perimeter and area of a shape change when it is scaled by a factor of 3?

Everyone eventually used either a square, rectangle or equilateral triangle. It seems that the easily formed grids most readily lead to insight into the effect on perimeter and area of scaling. However, the effects were worked out by counting rather than by multiplying, which reinforces multiplication as repeated addition rather than as scaling. Since multiplicative reasoning (see Thompson 2002) depends on multiplication seen as a scaling rather than simply as repeated edition, the fact that our awareness immediately directs us to repetition is of considerable concern.

How does the awareness generated by such tasks extend to other non-grid shapes? When asked to consider a non-grid shape, one person proposed enclosing a shape in a rectangle and then using the grid of rectangles. There might still be some work to do to see the requisite invariance of ratios, and again the underlying awareness is of multiplication as repeated addition.

There seems to be a subtle difference between knowing the rule for scaling areas, being aware of the nine-ness for scaling by three, seeing nine-ness as a direct perception rather than revealed as the result of counting, and making use of multiplication.

I had intended to then ask people the map-scalers question:

> Two people each have identical copies of a single map of the locality where they are. However the two people are at different places on their maps. Each person scales their copy of the map be the same scale factor, but using their own location as their centre of scaling. What will the difference in their scaled maps?

The fact that there is no difference is either a basic awareness which informs all the 'seeing' of shapes as being similar, or the result of the application of Thales theorem, in which a triangle cut by a straight line parallel to one side produces a scaled (similar) triangle, and vice versa.

The point is that having different centres of enlargement makes no difference to the result, which I find to be simultaneously obvious (a direct perception) and a surprise (things close to one centre might conceivably be scaled differently when the other centre is used).

The question arose as to the relationship between *core awareness* and *intuitions*. If you take a Fischbein (1987) approach to intuition, then core awarenesses may at best overlay prior inappropriate intuitions. Sometimes though, intuitions can be displaced by more informed intuitions and awarenesses. A desirable state of affairs would be that intuitions are taken as plausible conjectures to be tested in action, but this is far too generous a proposal: human beings often act on the basis of intuitions acting as awarenesses, that is, as the basis for action. Since awarenesses, as the bases of action, may or may not be appropriate bin any situation, an broad use of 'intuition' would be the same as 'awareness'. However I want to use awareness as something that can be educated. Indeed, following Gattegno (1987), 'only awareness is educable'! This conflicts with Fischbein's sense of intuition as being somehow below the level of modification. The awarenesses I am interested in bring to articulation the bases for action rather than merely prompts to action, as might be expected from intuitions. Expert awareness is informed by integrated practices in the form not only of actions available to use, but also reasoning associated with the actions, including justification, how it works, and inner incantations and kinaesthetic tendencies which guide the actions.

The question also arose as to whether *core awarenesses* are fundamental for the individual, or fundamental for mathematics. It might be useful to distinguish between the idiosyncratic and the mathematically canonical. It is certainly the case that mathematicians talk and write as if there are canonical awarenesses which provide the foundation for established mathematical concepts, theorems and techniques. However, because they are so hard to articulate, they rarely appear explicitly in texts.

Task 2

In the two diagrams below (figure 1), all the component figures are squares. Could the overall rectangles be squares?

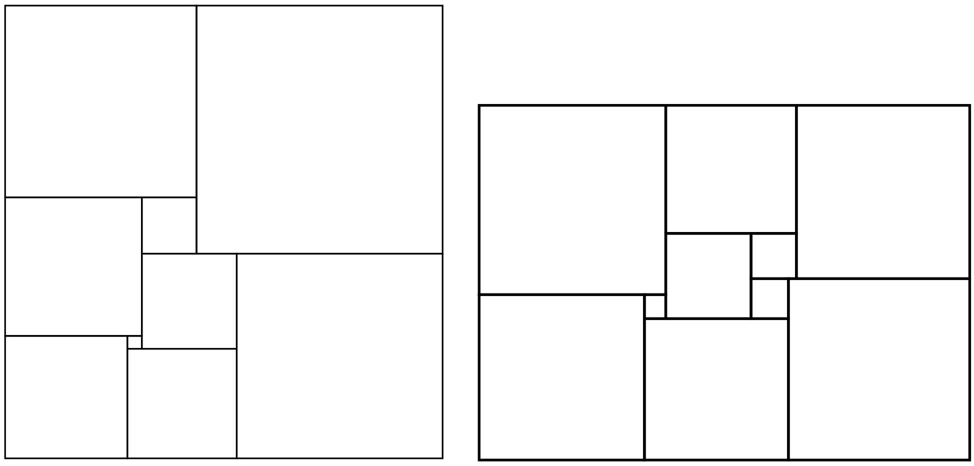

Figure 1: Task 2

Several people started by assigning letters to the sides of the squares around the outside, saw that they could, if pressed, write down multiple relationships, and gave up. Others persisted in trying to see geometrically rather than resorting to algebra. In the discussion, attention was directed to the smallest square, and talked about as a possible unit, but no-one that I was aware of assigned it a value of 1, proceeded to assign the square to its right a side length of x and then developed the sizes of all the others, ending up with a sequence of reasoning showing that the left hand figure could never be a square. Making use of the property of a square that all its edges are equal by instantiating it or applying it over and over again in sequence leads to the development of the sizes of the squares.

Being aware of possibilities of working either inwards from the outside or outwards from the inside can arise through taking the time to gaze: to hold the whole for a time so that you get a sense of the whole, and perhaps find a choice of actions becoming available, rather than diving in to the first thought that comes.

I recently encountered a logo for a cycle company (figure 2). At first it meant nothing to me, but while preparing for the session, I asked myself whether all the rectangles could have the same edge-ratio:

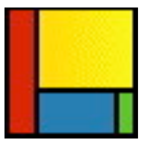

Figure 2: Logo for cycle company

and it turns out that this extra condition forces the edge-ratio to be $\sqrt{1+\sqrt{2}}:1$. Notice that I already had experience of square-development tasks, so this presumably provided both an awareness to action, provoking the asking of the questioning the first place, and a confidence base in how to approach it.

Task 3

Say What You See in the following diagrams (figure 3)

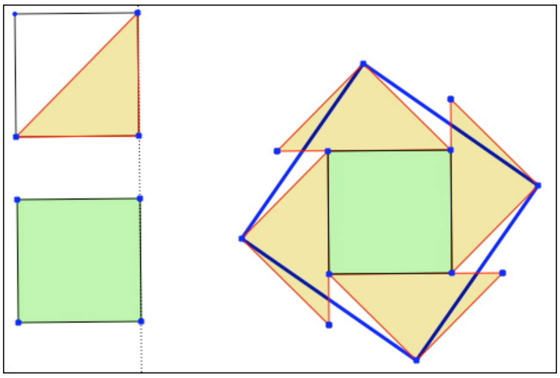

Figure 3: Say what you see

It was immediately assumed (as intended) that the two squares on the left were the same size, and that the same shading in different places implied a copy. Many people quickly reckoned the large tilted square to be three times the smaller central square, assuming that the bits of triangles sticking out exactly matched the unshaded bits inside. In other words, squares and shaded triangles were discerned, relationships between shadings recognised, and possible relationships between sticking out and as-yet-unshaded regions. The tilted square was discerned, and identified as instantiating the property of being a square. A conjecture arose and led to the conclusion that the edge length would be √3.

This still left open the question of whether the √3 was justified. The triangles sticking out and the triangular shapes inside certainly have the same angles and so are similar, due to the shaded triangles being isosceles right triangles. Why the tilted square bisects the segment of the hypotenuse of each of the shaded triangles still has to be reasoned. Eventually attention might be attracted to what is known … that the shaded triangles are isosceles and so their two arms are equal, giving a pair of equal sides in the pairs of similar triangles, making them congruent.

Here we see, or experience, the movements of attention triggered by awarenesses and by shifts in how we attend, leading to the behaviour we know as geometrical reasoning. It is entirely unreasonable to expect people at a BSRLM day to be relaxed enough to locate such a chain of reasoning in the middle of a session. Some people tried to rotate the shaded triangles into the tilted square but encountered an overlap which was difficult to deal with. They wanted a diagram like the familiar √2 diagram, and this desire directs attention along a chreod or valley-like channel (Waddington 1977) that may not be fruitful. Nevertheless it is sometimes a useful strategy to follow perceived similarities with past experience.

The figure is a snapshot from a dynamic geometry screen which permits the shaded square to be a different size to the square with the shaded triangle. Does the construction still 'work'? Such a question involves a shift of attention from what 'is', to what 'could be', conjecturing the possibility of a more complex relationship, indeed a property of which this figure is a special case. The reasoning carries through because the size of the shaded square was never used. So the tilted square is the sum of the shaded square and twice the half-shaded square.

Observations made during the session included the ease with which we can get trapped in a chreodic line of thinking that is hard to jolt oneself out of. The longer it is pursued, the higher the walls on either side (or the deeper the ruts), and so the greater the energy required to 'tunnel through', in the quantum sense.

My intention had been to highlight the way that attention shifts from one thing to another, and also the way sometimes you are gazing (holding a whole), however briefly; sometimes you find yourself discerning details; sometimes recognising relationships between or among discerned details; sometimes perceiving a property which might be instantiated as a relationship between discerned details; and sometimes reasoning on the basis of (agreed) properties. These for me form distinguishable states, which may occur ever so briefly, or which may persist for a period of time (see Mason 2003, Mason and Johnston-Wilder 2004).

These movements of attention provide clues as to underlying awarenesses which enable the actions to take place. For example, the dead-end experienced by some in the square development task highlights awareness of algebraic possibility combined with some sort of aesthetic or value judgement about the worthwhile-ness of pursuing it, but without sufficient motivation to try another approach. The original scaling task brings to the surface the discrete basis for scaling which somehow has to be extended to continuous scaling by non discrete scaling factors, while the map-scaling problem directly highlights an important awareness.

References

Fischbein, E. 1987. *Intuition in Science and Mathematics: an educational approach.* Reidel: Dordecht.

Gattegno, C. 1987. *The science of education part I: theoretical considerations*. New York: Educational Solutions.

Mason, J. 2002. *Researching Your own Practice: The Discipline of Noticing*. London: RoutledgeFalmer.

Mason, J. 2003. Structure of Attention in the Learning of Mathematics. In *Proceedings, International Symposium on Elementary Mathematics Teaching* ed. J. Novotná, 9 – 16. Charles University, Prague.

Mason, J. and S. Johnston-Wilder. 2004. *Fundamental Constructs in Mathematics Education*. London: RoutledgeFalmer.

Thompson, P. W. 2002. Didactic objects and didactic models in radical constructivism. In *Symbolizing and Modeling In Mathematics Education* eds. K. Gravemeijer, R. Lehrer, B. van Oers, and L. Verschaffel, 191-212 Dordrecht, The Netherlands: Kluwer.

Waddington, C. 1977. *Tools for Thought*. London: Jonathon Cape.