The role of context in linear equation questions: utility or futility?

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It is common practice in Key Stage tests and GCSE Mathematics to embed mathematics in real-world contexts. However, the practice has been criticised by some researchers on the grounds of artificiality and construct validity. This paper considers the role of context in four linear equation questions, concluding that the purpose of the context is not utility, but concept formation and abstraction.

Keywords*:* Mathematics; Assessment, Real World Context, Word Problems, Linear Equations

1 Introduction

It is widespread practice, both is classroom tasks and in more formal, summative assessments to present mathematical tasks in real-life contexts. The use of contextualisation reflects the trend internationally towards assessing 'mathematical literacy', defined in the PISA framework as

> … an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen' (OECD 2005, 72).

If, as Freudenthal (1991) and the advocates of realistic mathematics education propound, the roots of mathematical understanding lie in mathematising real-world contexts and thereby engaging in a process of guided re-invention, it would seem natural that assessment of students' understanding should equally place mathematics firmly in real-world contexts.

However, an increasing body of educational research has been ambivalent about the effects of contextualising mathematical tasks. While some of this research (Vappula and Clausen-May 2006) has pointed to the beneficial effect a context can have for pupils, by providing them with a 'model to think with' (Clausen-May 2005), others have pointed to the possibilities of misinterpretation of the task when presented in a context (Silver 1993, Verschaffel L. 1994, Cooper and Dunne 2000, Wiliam 1997).

Criticism of the role of context in compromising the validity of test items is not confined to mathematics tasks: in a recently published study, Ahmed and Pollitt (2007) tested a number of science tasks designed for 14-year-olds, in which the context was manipulated to provide varying degrees of 'focus', and found that the items' test validity, or 'construct fidelity', as they call it, differed substantially.

In another recently reported study, Shannon tested three different contextual representations of a linear function task – stacking supermarket carts (trolleys), shopping baskets and paper cups – and found differences in facility levels (Shannon 2007). These she explains not in terms of the familiarity of the items used to the task solvers, but by analysing the ease with which the salient features of the geometry of the stacking diagram could be abstracted into the variables required by the mathematics. She claims that the modelling of everyday objects with mathematics as a motivational tool is relatively unimportant in these tasks, compared to the opportunities they provide for mathematical abstraction and justification.

Presenting mathematical tasks in everyday contexts has a long history. A well known problem from the Rhind (or Ahmes) papyrus of 1650 BC (Boyer, 1985) may be stated in the form:

> 7 houses have 7 cats per house. Each cat catches 7 mice. Each mouse eats 7 ears of spelt, each ear would have produced 7 hekats of grain. What is the sum of the houses, cats, mice, ears and grain?

Word problems in mathematics are obviously not a modern invention! Many have criticised the artificiality of such problems on the grounds that they encourage an image of the utility or applicability of mathematics which is essentially false. Clearly, in the Egyptian example, the underlying mathematical model is completely unrealistic; moreover, the answer requested is equally useless, providing no useful information about the context of the question.

It would of course be pedantic to criticise this harmless, whimsical trifle of a problem on the grounds of a lack of utility or sound real-world modelling! Is this really a 'Macguffin' (Wiliam 1997) – an example of 'Maths looking for somewhere to happen'? Does their artificiality lead students into reacting against notions that mathematics might be genuinely useful? Clearly not: this is manifestly a puzzle, a dressing-up of a simple arithmetic question in a hopefully attractive or amusing form. One might just as well criticise a sudoko puzzle for being useless. The success of the problem depends not on realism, but upon whether the context captures the interest of the solver.

It is clear from the above discussion that the relationship between context and the mathematics which it embodies or represents is a complex one. This paper explores this relationship in more detail, by considering the types of context utilized in an exercise from a current textbook (Hanrahan et al. 2004) on formulating and solving linear equations.

2 Some questions on linear equations.

Consider the following four questions (Hanrahan et al. 2004, 10).

> 1. In 18 years time, Halley will be five times as old as he was 2 years ago. How old is he now?
>
> 2. A train has 8 coaches, f of which are first class and the rest standard class. A first class coach sits 48 passengers, a standard class 64. The seating capacity of the train is 480. How many standard class coaches does it have?
>
> 3. Joe buys 18 kg of potatoes. Some of these are old potatoes at 22 p per kilogram, the rest are new ones at 36 p per kilogram.
>
> (i) Denoting the weight of old potatoes he buys by w kg, write down an expression for the total cost of Joe's potatoes.
>
> (ii) Joe pays with a £5 note and receives 20p change. What weight of new potatoes does he buy?
>
> 4. The largest angle of a triangle is six times as big as the smallest. The third angle is 75°. Find the size of the three angles.

Question 1 will be familiar in type to all students of mathematics, and uses the context of age to pose a question capable of formulation and solution by a linear equation. To claim that this puzzle represents a genuine application of mathematics is, as in the Egyptian example, far-fetched. Why should we want to know John's age? Why doesn't he know it in the first place? If necessary, we could always look up his birth certificate! The question is whether we should banish such problems, and look for more 'genuine' examples of mathematical modelling. Surely there must be contexts out there which are less artificial, more genuine, and hence (we assume – but let's come back to this) – more motivating?

Is question 2 any better? Well, I suppose it is slightly more feasible that there exists a compiler of trains who may wish to form a train of 8 coaches with exactly 480 seats (how convenient is that!). But our 'fat controller' is more likely to be concerned with the number of coaches she has available since the 8.50 a.m. from Paddington broke down outside Reading! Instances of linear equations being utilised in the logistics of the railway industry are, I predict, pretty rare, if not completely non-existent.

Question 3 exploits the familiar everyday context of shopping, and endeavours to situate the context more realistically by proffering irrelevant information that Joe 'pays with a £5 note and receives 20 p change', rather than stating that he paid £4.80 for his potatoes. However, the problem is no less artificial, since his receipt would no doubt itemise his purchases, and Joe is no more likely than our 'fat controller' of question 2 to deploy algebraic methods to recover this information.

In question 4, the context here is purely mathematical, so any semblance of utility must be in terms of the perceived significance of the geometrical result. In this case, this is negligible. But is the question actually more acceptable for having no pretence towards real-life application? Perhaps is it more palatable to construct pure questions in a purely mathematical context? It does appear that the harder that these questions attempt to simulate reality, the easier it is to uphold charges of 'McGuffinism'!

All these questions, and indeed all of the questions in the exercise, may be regarded as 'applications' of linear equations, in the sense that they involve formulating and then solving them. However, they have no practical utility value. None of the results provide significant information about the context. They are exercises – opportunities to flex linear equation solving muscles. The only utility they might have is to develop these muscles, muscles which are essential for developing general mathematical skills, which, in turn, might be useful for solving unspecified problems in future.

Let us return to the 'McGuffin' question. Are these questions examples of 'maths looking for somewhere to happen'? It seems to me that this depends upon the plot, or our pupils' understanding of the purpose of the problems. If we claim some utility for them, simply because they employ recognisable real-life objects like 'trains', 'ages' and 'potatoes', then we are creating a mirage, selling a lie. If we are more honest, and treat them as nothing more than exercises in formulating and solving equations, then the charge of 'McGuffinism' is less well founded.

If there is, indeed, little utility in these questions, then how do we answer the inevitable questions, such as:

> 'Why are we doing this, miss?'
>
> 'When am I ever going to have to solve equations like this anyway?'

> 'Everyone says maths is supposed to be so important, so why is it so useless?'

This returns us to the issue of motivation alluded to before. Clearly, solving these questions is not motivated by utility, and equally their utility is not motivating. So, the idea that employing context to a question automatically enhances its motivating power is simplistic. I shall return to this issue later.

3 Contexts as models for thinking with

Some researchers on context (Clausen-May 2005) have pointed to the potential role of contexts as 'models to think with'. How would this idea relate to our three problems?

Take the first problem, and let's think it through with the context. We could solve it like this, using a simple tabulation:

Age 2 years ago	Age now	Age in 18 years time	5 times?
0	2	20	No
1	3	21	No
2	4	22	No
3	5	23	No
4	6	24	No
5	7	25	Yes

It is debatable whether this method takes any longer than the algebraic one. If you spot that the age after 18 years must be a multiple of 5, it is probably quicker!

What about using the context in the second problem as a 'model to think with'? The following solution is suggested. We need 480 seats with 8 carriages. If they are all first class, then this would give us 8 ´ 48 = 384 seats, which is 96 too few. Replacing each first class carriage with a second class gives us 16 more seats. 96 divided by 16 is 6, so replace 6 first classes with seconds gives the answer: 2 first and 6 second! No algebra required here, just some clear thinking skills.

A similar approach is suggested by the context in question 3. If all 16 kg were old potatoes, the bill would be 18 ´ 22 = £3.96, which is 84p less than Joe paid. As new potatoes are 14p more expensive, he must have bought 84 / 14 = 6 kg of new potatoes. Again, no algebra is required!

What about question 4? Well, I've got 180° in my triangle; one angle takes up 75°, leaving 105° for the other two. One is 6 times the other, so I need to divide up 105 in the ratio 6 to 1. 105 divided into 7 shares is 15°, so one is 15° and the other is 90°.

Whether these solutions are harder or easier than the algebraic ones is debatable. The point here is that they are all quite different, whereas the algebraic method is structurally identical in each case! Using the context as a 'model to think with', at least in the ways proposed above, would appear to side-track the use of the algebraic methods intended by this exercise. Of course, as these examples come from an exercise which follows examples in which linear equations are formulated, it is perhaps less likely that 'ad hoc' methods will be adopted by students. However, in my experience, it is quite common for students faced with, for example, a GCSE question which requires algebraic formulation, will resort to first principles thinking within the context (and, sometimes, with some success).

In fact, far from requiring our students to utilise the context as a 'model to think with', in practice we ought to be encouraging them to think without the model, in other words to abstract the common patterns and elements from these contexts, in order to formulate a more general strategy which can be applied to all the problems in the exercise 'without thinking'. This is precisely the horizontal and vertical mathematisation of Treffers (1987). The utility of the contexts in these questions is that they serve in the reification (Sfard 1991) of the concept of the algebraic solution of linear equations.

4 'Why are we doing this miss?'

Returning to the motivation issue, students rightly reject the claim to real-life utility of word problems in the mathematics classroom. Their behaviour in misinterpreting such problems has been exhaustively researched. However, this behaviour has perhaps itself been misinterpreted, since the goal of context in these problems is not realistic modelling – *models* for – but mathematical concept formation, or providing *models of.*

How do we answer the question of motivation truthfully? We should first abandon any notion that these questions have any merit as examples of mathematical utility, and encourage students to reflect on the variety of contexts which our newly honed tools – or tools in process of being honed – can be applied. Encouraging students to reflect on their own learning process and reinforce the pattern recognition by making up their own examples (no doubt equally artificial) help to focus on this.

Of course, we all know that mathematics is actually incredibly useful – that mathematical modelling is one of the most powerful tools for analysing the real world developed by humankind. However, to confuse the process of manufacturing mathematical tools with mathematical modelling would be like taking a half-assembled car out of a production line and attempting to drive it away without a gearbox!

This paper has discussed only four contextualised questions, and it is possible that these questions are not fair representatives of questions which may have genuine modelling aspirations. Some researchers into word problems (Greer, 1997) have advocated introducing more elements of realism into classroom tasks, for example by adding irrelevant information, which then has to be discounted by the solver. However, expecting students to engage in genuine mathematical modelling activity before they serve an apprenticeship in formulating algebraic equations, and learning abstract, analytical methods for solving them, is perhaps itself unrealistic. Many students find the process of translating real-life numerical concepts into algebraic variables demanding enough, without being deflected by realistic 'noise'.

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