# A constructionist approach to mathematical generalisation

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The work presented here arises from the MiGen project, which aims at designing, building and evaluating a pedagogical and technical environment for improving 11-14 year-old students' learning of mathematical generalisation. This paper presents the preliminary analysis of the initial data collected in 24 sessions with year 6 and year 7 students. During these sessions, a prototype microworld was used to explore the range of functionalities we require, and collect initial data regarding students' strategies and potential prompts that the system could ultimately provide to both students and teachers. A key design objective is to develop an environment that makes it as natural as possible for students to express generality, and a means to do so, rather than simply encourage them to consider special cases, or spot patterns.

## Introduction

Accommodating the idea of mathematical generalisation, recognising and analysing pattern and articulating structure are key aspects of mathematical thinking and a central rationale for algebraic expression. These have always been elusive ideas for students, who routinely find themselves unable to understand what algebra is all about. Even though students are able to identify and predict patterns (Mason, 2002), they are unable to articulate a general pattern or relationship in either natural language or –more demanding still– in algebraic symbolism (Hoyles and Küchemann, 2002). What is required therefore, is to design situations that are rich in affording opportunities for the construction and analysis of patterns, and provide both a rationale and computational support for expressing generality.

Our aim is to design, build and evaluate a pedagogical and technical environment for improving 11-14 year-old students' learning of mathematical generalisation. This system consists of a microworld that supports students in their reasoning and problem-solving of generalisation tasks. Two further components aim to provide personalised support adapted to students' construction processes and foster an effective online learning community by advising learners and teachers as to which other students' constructions could fruitfully be viewed, compared, critiqued and built upon.

This paper describes some design principles underlying a prototype of the microworld (called 'ShapeBuilder') that was built to explore an emerging set of functionalities we would implement in the technical environment. We discuss how the major features of the chosen generalisation task were deployed and the design of the pedagogic environment developed. Small-scale pilot studies were carried out in 24 sessions with individual students or groups of year 6 and year 7 students. Finally, the preliminary analysis of this data is discussed.

## Theoretical background

Moss and Beaty (2006) claim that generalising problems are not themselves the cause of students' difficulties, but rather the way they are presented and the constraints of teaching approaches. In school, a standard approach leads towards pattern spotting, where students are taught techniques of finding the $n^{th}$ term by comparing consecutive terms usually presented in tables, and divorced as soon as possible from any structural features of the object being modelled. The difficulty is that this approach tends to generate correct answers (and therefore achieve requisite performance on tests) but fails to generate a motivation for generalisation, and, more importantly, does so by encouraging a breakage between symbolic (or arithmetic) representation and how it is represented (Hoyles and Healy 1999).

Emphasizing the numeric aspect of patterning tends to lead to the variables becoming obscured (Noss et al. 1997; Noss and Hoyles 1996) and students' ability to conceptualise relationships between variables, justify the rules and use them in a meaningful way becomes limited (Moss and Beaty 2006). Teachers tend to teach "the abstracted techniques isolated from all context" or alternatively "the technique as a set of rules to be followed in specific contexts" (Sutherland and Mason 2005) to help their students find the rule. Since students have, according to Mason (2002), natural powers to generalise, this could result in students' own powers atrophying due to lack of use.

A further difficulty faced by secondary school students is the use of letters to stand for unknowns (see Küchemann, 1981 for seminal work in this regard). They struggle to grasp the idea of letters representing any value (Duke and Graham 2007) and tend to lack the mathematical vocabulary needed to express generality. Even though it is a reasonable strategy to introduce algebra early, there is still the issue of how to introduce it so that students can make the transition from simple arithmetic to algebra. Other researchers (Warren and Cooper 2008) report how students' written responses lack precision, which supports the view of primary school students' inexperience with mathematical language. Even if students succeed in expressing generality, their first attempts, naturally enough, are in natural language. Motivating an alternative way (and one that is seen by the students as more effective) is a crucial concern (see Redden 1996).

Taking all these concerns into account in the design of the microworld and the associated activities, we could encourage students to write expressions in a general form in addition to giving an imprecise description in words. This articulation process needs to be addressed explicitly so that students could learn to express their thinking using algebraic notation. Indeed, a major rationale for designing with digital technologies is allowing students to see different representations, such as symbolic, iconic, numeric or even verbal ones, and realise the relationships and the equivalence of different representations. Following a constructionist pedagogical approach (e.g. Papert 1993), students are encouraged to construct entities and relationships using appropriate quasi-algebraic notation.

## Methodology

A prototype of the microworld, ShapeBuilder, has been developed to help us inform the design of our system and evaluate the effectiveness of our pedagogical and technical approach. Throughout the development of ShapeBuilder, we have followed an iterative design process, interleaving software development phases with small-scale pilot studies with individual or groups of students of our target age (11-14 years old). We have also integrated feedback from teachers and teacher educators as well as the students who participated in the studies. Twenty-four sessions with students, during which their teacher was present, were carried out and comprised two activities. The first activity introduced the functionalities of the prototype microworld through the use of progressively more difficult tasks and aimed at familiarising students with the features of the system and its affordances. The second activity involved the 'pond-tiling' activity. The simplest version of which can be described as follows: "Given a rectangular pond with an integer length and breadth, how many 1×1 tiles are needed to surround it?". We have focused on pond-tiling because we feel – at least in the final iteration of design - that we have created a solution space that exploits computational power, and it naturally lends itself to a variety of different representations, which could be described with the use of different yet equivalent expressions for the number of tiles required. This allows students to explore their own solutions, and compare them with others', raising the necessity (or at least encouragement) to represent the relationships semi-formally.

## Students' interactions with Shapebuilder

ShapeBuilder allows students to build various shapes (in the first instance, rectangles). Its main aim is to encourage structured algebraic reasoning of KS3 students by providing appropriate facilities for them to build shapes *using expressions* derived from special-purpose tools. Once a shape is defined, the student can move it, attach it to other shapes and alter its expressions as desired. An important feature of the software is the ability to define expressions using properties of the constructed shapes. For example, the student is able to obtain an icon-variable which evaluates to the current value of that dimension of the shape. These icon-variables act as an intermediate language for students and ease the transition from a specific construction, and therefore expression, to a general construction represented by a general rule.

ShapeBuilder allows students to construct their own models, explore them and, with some guidance from the teacher or the researchers, succeed in constructing more general models. After some introductory activities, students explored the pond-tiling activity. They usually started by building specific constructions using numerical values for the surrounding tiles of a pond. A typical intervention, inspired by previous research in dynamic geometry (Healy et al 1994), was challenging the students to construct a solution that is impervious to "messing-up", that is a construction that would be valid even if the values for either the width or the breadth of the pond were changed. This proved of great importance for students' progress towards generalisation. Students were surprised to see that their construction was messed-up and were trying to figure out why.

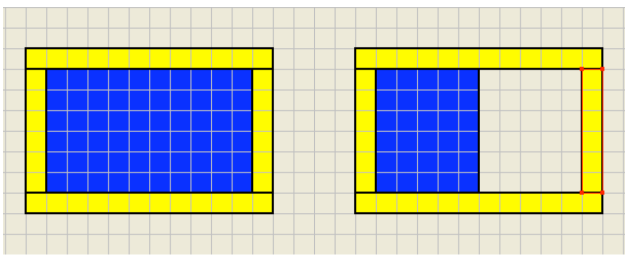

Figure 1. 'Messing-up' a student's construction

For example, Jack, a year 7 student had constructed a pond and its surrounding tiles using specific values. His pond was of width 10 and breadth 6 and he surrounded it (figure 1) by using two 1×6 rectangles (vertical) and two 12×1 rectangles (horizontal). He was then surprised to see that his surrounding didn't work for half the size of his original pond.

Researcher: What would it [the width of the pond] be if it was half?

Jack: Five.

Researcher: So, now that it is five, how many [tiles] are needed?

Jack: The width plus … six. I think.

Teacher: You made this one, half as big?

Jack: I think I've done this one wrong.

This 'messing-up' strategy led students to question their choice of specific values and encouraged them to use and appreciate the use of icon-variables, and, as far as we can tell, think in terms of these general representations rather than specific measurements and therefore reach some form of mathematical generalisation.

The next step was for students to find a rule for the number of the surrounding tiles and name their expressions. They usually came up with names like 'width of pond' or 'number of yellow tiles'. This articulation process could lead to the use of single words or even letters once students were comfortable with what they represent. Such a process could, we hope, initiate their transition to the use of letters in the traditional algebraic symbolism.

Another approach that proved of great value for students' efforts to generalise was collaboration. During the pilot studies, some students were interviewed in pairs and even though they worked individually till they reached a general rule, afterwards they were grouped and prompted to discuss their rules. This activity forced them to articulate their way of thinking and their strategies that helped them reach that rule. It also allowed them to help each other understand the existence of different, but equivalent solutions. An example that proved the collaboration's importance was that of 2 year 7 students, Amy and Jane. Their rules are shown below (Figure 2):

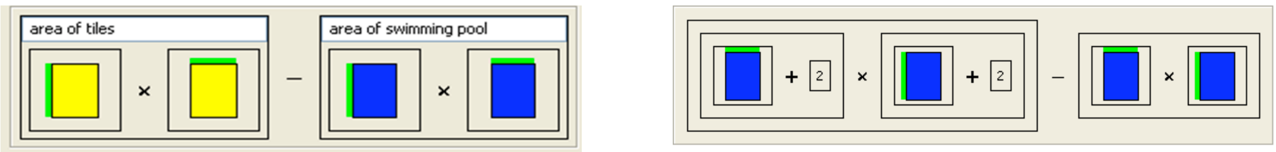


Figure 2. Amy's (left) and Jane's (right) rules.

Jane: I did the… like you did… the height of the pond plus two and then the width of the pond plus two. And then I did...

Amy: that wouldn't work… […] If you did the height of the pond plus two and then the width of the pond plus two… you don't need the width of the pond plus two… because otherwise you would have like…

Jane: No, I know, but it does work. I don't know. I thought that, but it actually does work if we make the shape… somehow.

After arguing about these seemingly different solutions, students were led to another solution which was accepted by both as correct. They agreed that doubling the width and the breadth of the pond and adding four more tiles for the 4 corners gave them the correct number of the surrounding tiles. In the end, they were satisfied with their final solution as it represented a shared solution strategy. This incident illustrated the advantages of tightly-constrained (i.e. designed) collaboration that led students to co-construct an acceptable and more comprehensive for them solution.

## Conclusion and future work

Earlier, we gave emphasis to three aspects that provide support to the students. First, icon-variables, because they represent any attribute of an object, are considered a source of expressing situated abstractions, in that generality can be rigorously expressed with them, but without recourse to standard algebra. We recognise this is a tentative first step towards constructing a language of expression, and we are currently considering how to extend the idea. Second, the “messing-up” strategy challenged students’ comprehension of the pond-tiling activity and directed their attention to ill-formed attempts at generalisation as well as providing a rationale for generality. Third, our careful design of collaborative interaction seems to have borne fruit in generating an appreciation of the correctness and the generality of students’ rules. It seems that co-construction (either of shapes or expressions) has the potential to encourage students to come up with an explicit strategy that they have to share with the other students as well as a language to describe it. As has been observed in previous research (e.g. Noss and Hoyles 1996), cognitive conflict with other students or reflection upon one’s own actions in the light of differing points of view is a way to foster the development of mathematical meaning.

Although we cannot elaborate it here, we conclude with a mention of our intention to develop computationally-intelligent tools that support students and teachers during interactions with the microworld. Our goal is to provide assistance to students and report back to the teacher issues regarding the students’ progress, reveal which students need further assistance and alert teachers of students’ possible misconceptions. This is a challenging agenda both in terms of designing the necessary artificial intelligence, and in finding the right ways to represent information to the teacher (and, of course, the student).

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