Learners' Understanding of the Hierarchical Classification of Quadrilaterals

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The hierarchical classification of quadrilaterals might be regarded as an area of study which would help to promote the development of geometrical thinking. This paper reports our investigations in this topic discussed in our former study (Fujita and Jones, 2007, RME vol. 9). In particular, by synthesising past and current theories in the teaching of geometry (van Hiele's model, figural concepts, prototype phenomenon etc.), I propose a theoretical model and method to describe learners' cognitive development of their understanding of hierarchical relations of quadrilaterals. I also consider how the model could be utilised to describe and analyse learners' understanding of the hierarchical classification of quadrilaterals by using pilot data collected in 2008.

Keywords*:* The learning and teaching of geometry; Hierarchical Classification of Quadrilaterals; Cognitive development; Image and concepts

Introduction

The teaching of geometry provides not only a key vehicle for developing learners' spatial thinking and visualisation skills, but also a major opportunity to develop their ability in deductive reasoning and proving (Battista 2007; Royal Society 2001). The new National curriculum for the Key Stage 3 and 4 in England (QCA 2007) now specifies geometrical content 'Geometry and measures', not calling this area of mathematics 'Shape and space', and geometrical reasoning is regarded as one of the central mathematical processes which all students are expected to develop.

The hierarchical classification of quadrilaterals might be regarded as an area of study which would help to promote the development of geometrical thinking (Fujita and Jones 2007). To reason successfully about the answer to the question about whether a rhombus is a (special type of) parallelogram, learners need not only to be able to control its image, but also to examine properties (concepts/theorems), and we consider this is a good exercise to promote learners' geometrical thinking. However, the study of the hierarchical classification of quadrilaterals is a classic case in which many learners have difficulties to grasp, e.g. 'A square is not a parallelogram because parallelograms are slanted', and our research focuses on this matter.

This paper reports our investigations in this topic discussed in our former study (Fujita and Jones 2007). In particular, we further propose a theoretical model and method to describe learners' cognitive development of their understanding of hierarchical relations of quadrilaterals. We consider how the model proposed in this paper would be ustilised to describe and analyse learners' understanding of the hierarchical classification of quadrilaterals by using our pilot data collected in 2008.

Theoretical framework: 'Q-Level' for the understanding of parallelograms

We employ the following theories to give a comprehensive explanation why the hierarchical classification of quadrilaterals is difficult for many learners, and how we could deal with this problem: van Hiele's level of geometrical thinking (Crowley 1987; van Hiele,1999) and Koseki's level of the understanding of the hierarchical understanding of parallelograms developed in Japan (Koseki 1987); Tall and Vinner's concept definition (Tall and Vinner 1981), Fischbein's figural concepts as the nature of geometrical figures, (Fischbein 1993) and personal and formal figural concepts (Fujita and Jones 2007); Hershkowitz's prototype phenomenon of geometrical figures (Hershkowitz 1990). Considering these theoretical discussions, we assume that learners' understanding of the hierarchical relations between quadrilaterals would develop through certain levels, and adopt/modify a model proposed by Koseki (1987; See also Fujita and Jones 2007). To avoid confusion between the levels of van Hiele's model, we use 'Q(uadrilateral)-Level' which specifically focuses on the development of the hierarchical relation of quadrilaterals, summarised as follows.

- Q-Level 1-i) for learners who do not have basic knowledge of parallelograms
- Q-Level 1-ii) for learners who have very limited figural concepts of parallelograms
- Q-Level 2-i) for those who began to extend their figural concepts, (for example that rhombuses are also parallelograms) but still do not have sound understanding
- Q-Level 2-ii) for those who can accept squares, rectangles and rhombuses are also parallelograms but not fully grasp relationship between, for example, squares and rectangles
- Q-Level 3-i) for those who begin to have formal figural concepts of parallelogram
- Q-Level 3-ii) for those who have formal figural concepts of parallelogram with the understanding of class inclusion.

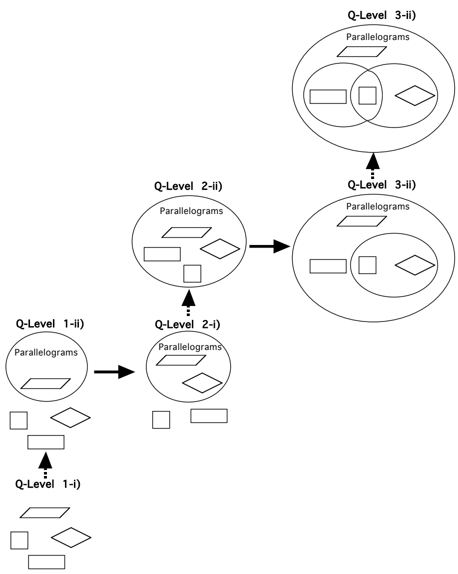


Figure 1: Development of the Q-Level

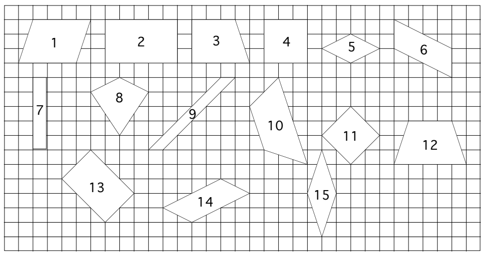
In addition, we consider four inter-related components 'images', 'definition', 'concepts' and 'reasoning' to capture this complex process of the development. For example, a learner who can state a correct formal definition of parallelograms, but has 'slant' images for parallelograms and believe that a statement 'parallelograms don't have right angles' is true would be in Q-Level 1-ii) for 'image' and 'concepts' and Q-Level 2-ii) (or above) for 'definition'. Through the development, images and concepts would be harmonised, and learners would be able to use them flexibly to solve various problems in geometry.

Methodological proposal: How do we measure the Q-Levels?

Our preliminary research instrument is a research questionnaire based on the studies developed in Japan, in particular by Koseki et al (1987), Nakahara (1995), and Okazaki (1995). We chose to use a questionnaire as the first method as this is a quick and efficient way to collect and analyse data. In this section, we shall propose how we measure learners' Q-Levels with examples taken from our pilot study conducted to nineteen learners of their first year of the BEd study at a university. Nineteen is a relatively small number, and we are aware that the sample is still limited. The purpose of the use of this sample is not to generalise the result but to show how our research instrument is utilised to describe learners' understanding of hierarchical relations of quadrilaterals.

The first part is mainly checks learners' knowledge and understanding of parallelograms.

Q1.



Which of the quadrilaterals 1-15 above are …
(a) members of the Parallelogram family ( )
(b) members of the Rhombus family ( )
(c) members of the Rectangle family ( )
(d) members of the Square family ( )
Q2. What is a parallelogram? Describe it in words.
Q3. Read the following sentences carefully, and circle the statements which you think are correct.
(a) There is a type of parallelogram which has right angles.
(b) The lengths of the opposite sides of parallelograms are equal.
(c) The opposite angles of parallelograms are equal.
(d) There is a type of parallelogram which has 4 sides of equal length.
(e) Some parallelograms have more than two lines of symmetry.
Q4. Is it possible to draw a parallelogram whose four vertices are on the circumference of a circle? Choose your answer a. b. or c. If you choose a., state your opinion why it is not possible. If your answer is b., draw its shape and name.
a. No it is not possible, because …
b. Yes, it is possible.
c. I don't know

Figure 2: Q1-Q4 in the research questionnaire

While Q1-(a) and Q2 check learners' basic knowledge of parallelograms in terms of 'images' and 'definition', Q3 checks how they regard parallelograms in terms of the properties ('concept'). For example, if a learner regards a rectangle as a special type of parallelogram then s/he would choose the statement (a) as a true statement. Q4 ('reasoning') challenges them to determine parallelograms which can be inscribed in a circle (the answer is rectangles). Thus this question checks whether learners are able to use a hierarchical relationship to solve a problem. Table 1 summarises the marking criteria for each question from the questionnaire in figure 1. Each question is measured by the 'Q-Levels 1~2-ii)' described in the previous section.

	Q-Level 2-ii)	Q-Level 2-i)	Q-Level 1-ii)	Q-Level 1-i)
Q1-(a)	At least ten of the following: 1, 2, 4, 5, 6, 7, 9, 11, 13, 14, 15	At least six of 1, 5, 6, 9, 11, 14, 15 or eight of 1, 2, 4, 6, 7, 9, 11, 13, 14, 15	At least three of the following: 1, 6, 9, 14	Others
Q2	Correct definition & image	Stating too many properties	Statements involving limited images (such as 'slant rectangle')	Others
Q3	Correct for at least four of (a) – (e)	Correct for (b) & (c) + one of (a), (d) or (e)	Correct for at least one of (b) & (c)	Others
Q4	(b) and draws & names a rectangle	(b) and draws & names a square or a rhombus (b) and draws a correct image	(a) or (c) (if Q1 is Q-1-ii))	Others

Table 1: Marking criteria for Q1, Q2, Q3&Q4

Now, we shall show how these questions are utilised to measure learners' Q-Levels. Table 2 summarises detailed results from the nineteen learners, and this suggests some tendency of the development of the levels. For example, learners who achieved Q-Level 2-ii) for 'image' question (Q1) are likely to be at Q-Levels 2-i) or 2-ii) for 'property' question (Q3). Learners with Q-Level 1-ii) or 1-i) for 'image' are likely to be at Q-Level 1-ii) or 1-i) for Q3, 'property' question, e.g. the subjects 4, 7, 12, 15, 16 and 18. This also shows a typical case of prototype phenomenon, i.e. they choose prototype images of parallelograms and also considered there would be no parallelograms whose angles are all 90 degree. The result of Q2 is slightly better than the others and this suggests that the students could literally accept the concept definition of parallelograms. The result of the Q4 (reasoning) shows that the task is very difficult for a learner who has achieved Q-Level 2-ii) in 'images' and 'properties'.

Subject	Q1	Q2	Q3	Q4
1	2-ii)	2-i)	2-ii)	2-i)
2	2-i)	2-i)	2-ii)	1-ii)
3	2-ii)	2-ii)	2-ii)	2-i)
4	1-ii)	2-i)	1-ii)	1-ii)
5	2-i)	2-i)	2-i)	2-i)
6	2-ii)	2-ii)	2-ii)	2-i)
7	1-i)	2-ii)	1-i)	1-i)
8	2-i)	2-ii)	1-i)	1-ii)
9	2-ii)	2-ii)	2-ii)	2-i)
10	2-ii)	2-ii)	2-ii)	2-i)
11	1-ii)	2-ii)	2-ii)	1-ii)
12	1-ii)	2-ii)	1-ii)	1-ii)
13	1-i)	1-i)	2-ii)	1-i)
14	2-ii)	2-ii)	2-ii)	2-i)
15	1-i)	1-ii)	1-ii)	1-i)
16	1-i)	1-ii)	1-ii)	1-i)
17	2-ii)	2-i)	2-ii)	1-i)
18	1-ii)	2-i)	1-ii)	1-i)
19	2-ii)	2-ii)	2-ii)	2-i)

Table 2: Results from nineteen learners

The second part of the questionnaire checks the other relationships, squares/rectangles and rhombus/square. Due to the limited space, we shall introduce Q6 and Q8 which measure their understanding of 'definition' and 'properties' of rhombus and rectangle.

Q6. Read the following sentences carefully, and circle the statements which you think are correct. Also describe a rectangle in words.
(a) The lengths of the opposite sides of rectangles are equal.
(b) The opposite angles of rectangles are equal.
(c) There is a type of rectangle which has 4 sides of equal length.
(d) Some rectangles have more than two lines of symmetry.
A rectangle is ( )
Q8. Read the following sentences carefully, and circle the statements which you think are correct.
(a) The lengths of the opposite sides of rhombuses are equal.
(b) The opposite angles of rhombuses are equal.
(c) There is a rhombus which has right angles.
(d) Some rhombuses have more than two lines of symmetry.
A rhombus is ( )

Figure 3: Q6&8 in the research questionnaire

We consider a learner has a good understanding of, for example, the relationship between rhombus and square when s/he chooses correct images for rhombus (4, 5, 11&15 from the figure 1), and (c) and two from (a), (b) and (d) in Q8. We shall focus on the subjects 1, 3, 6, 9, 10, 14 and 19 here who achieved at least Q-Levels 2-i) or 2-ii) for Q1, Q2, Q3 and Q4 to consider whether they are Q-Level 3-i) or 3-ii). Table 3 summarises their performance of Q1, Q6 and Q8.

Subject	Q1-(b) (rhombus)	Q1-(c) (rectangle)	Q6 (rectangle)	Q8 (rhombus)	Q-Level
1	1, 3, 5, 6, 10, 12, 14	2, 4, 7, 11, 13	(a), (b), (c), (d)	(a), (b), (c), (d)	3-i)
3	10, 3	2, 13	(a), (b), (d)	(a)	2-ii)
6	No answer	2, 4, 7, 11, 13	(a), (b), (c), (d)	(c), (d)	3-i)
9	5, 11, 15, 4	2, 4, 7, 11, 13	(a), (b), (c), (d)	(a), (b), (c), (d)	3-ii)
10	5, 11, 15, 4	2, 4, 7, 11, 13	(a), (b), (c)	(a), (b)	3-i)
14	4, 5, 11, 15	2, 7, 9, 13	(a), (b), (c), (d)	(a), (b), (c), (d)	3-i)
19	10, 12	2, 7, 13	(a), (b), (c), (d)	(b)	2-ii)

Table 3: Performance of Q-Level 2-ii) learners

From this table, we can observe various, but complex process of the development from the Q-Level 2-ii) to 3-i). For example, subject 9 chose 4, 5, 11 and 15 as images of rhombuses and true for (a) – (d) in Q8, and 2, 4, 7, 11 and 13 as images of rectangles and true for (a) – (d) in Q6, which implies that s/he fully grasps the hierarchical relationship between parallelogram/rhombus/rectangle/square (Q-Level 3-ii)); subject 10 also understands rectangle/square relationship, but her/his understanding of rhombus is bit weak as this one considers 'Q8 (c) There is a rhombus which has right angles' as a false statement (Q-Level 3-i)); subject 14 did not choose 'squares' as rectangles (Q1-(c)) whereas s/he understands the rhombus/square relationship (Q-Level 3-i)); the subject 1 has a good understanding in general, but her/his knowledge of rhombus is a bit weak as s/he chose trapeziums as 'rhombuses' and so on.

Concluding remarks

In this paper, we proposed the theoretical model and method to describe learners' cognitive development of the hierarchical relations of quadrilaterals by using data from nineteen learners. In addition to collect more data by using the questionnaire, what we have to consider is obviously how we use this model to improve the current situation of the teaching of geometry. As the sample size is very small, we cannot draw any generalised conclusions, but it is worth considering the following points:

- From table 2, there might a tendency that learners could first grasp the formal definition of parallelogram, but they just 'literally' accept the definition. They would not be able to use this definition to accept/understand/deduce images and concepts of 'square as a special type of parallelogram'. The 'reasoning' would be very difficult to achieve as even Q-Level 2-ii) learners cannot control images and concepts to solve geometrical problems;
- The results might be able to identify weaknesses of learners. For example, table 3 suggests that learners in this sample group might have weaker knowledge of rhombuses. Such information would be useful to inform the design of learning environments for learners to improve their geometrical understanding;
- While the development from Q-Level 2-ii) to 3-ii) suggests that the process would be very complex, testing the questionnaire with a larger sample might be able to identify a common cognitive path.

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