BSRLM Geometry Working Group

WAYS OF LINKING GEOMETRY AND ALGEBRA: THE CASE OF *GEOGEBRA*

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This paper discusses ways of enhancing the teaching of mathematics through enabling learners to gain stronger links between geometry and algebra. The vehicle for this is consideration of the affordances of GeoGebra*, a form of freely-available open-source software that provides a versatile tool for visualising mathematical ideas from elementary through to university level. Following exemplification of teaching ideas using* GeoGebra *for secondary school mathematics, the paper considers current emphases on geometry and algebra in the school curriculum and the current (and potential) impact of technology (such as* GeoGebra*). The paper concludes by raising the implications of technological developments such as* GeoGebra *for the pre-service education and inservice professional development of teachers of mathematics.*

INTRODUCTION

Geometry and algebra are central to mathematics and have been called its "two formal pillars" (Atiyah, 2001). As such, school mathematics curricula around the world afford prominent positions to both geometry and algebra, especially at the secondary school level. In some countries this means entirely separate courses on geometry and algebra, while in other countries, the curriculum alternates topics from the various components of mathematics.

This paper discusses ways of enhancing the links between geometry and algebra through consideration of the affordances of *GeoGebra*, a freely-available open-source software package that provides a versatile tool for visualising mathematical ideas from elementary through to university level. Following a presentation of some examples of teaching ideas using *GeoGebra* for secondary school mathematics, the paper raises issues concerning the current emphases and treatment of geometry and algebra in the school curriculum and the current and potential impact of technology such as *GeoGebra*. Finally, the paper broaches the implications of all this for the pre-service education and inservice professional development of mathematics teachers as technology such as *GeoGebra* becomes more pervasive in mathematics classrooms.

SOFTWARE FOR VISUALISING MATHEMATICS

Within the host of software that is available to support the teaching and learning of mathematics, two prominent forms are *computer algebra systems* (CAS, which focus on manipulation of symbolic expressions) and *dynamic geometry software* (DGS, which concentrate on relationships between points, lines, circles and so on). Amongst the best-known examples of CAS are *Derive* and *Maple*, while *Cabri* and *Sketchpad*

are well-known DGS. As both these types of software have developed over the years, forms of CAS have begun to include graphing capabilities in order to help to visualise mathematics; likewise, DGS have begun to include elements of algebraic symbolisation in order to be useful for a wider range of mathematical problems.

A recently-development piece of software entitled *GeoGebra* (Hohenwarter, 2002; Hohenwarter & Preiner, 2007) provides a closer connection between the symbolic manipulation and visualisation capabilities of CAS and the dynamic changeability of DGS. It does this by providing not only the functionality of DGS (in which the user can work with points, vectors, segments, lines, and conic sections) but also of CAS (in that equations and coordinates can be entered directly and functions can be defined algebraically and then changed dynamically). These two capabilities are characteristic of *GeoGebra* in that the default screen (see Figure 1, with the language set to English) provides two windows in which each object in the algebra window corresponds to an object in the geometry window, and vice versa.
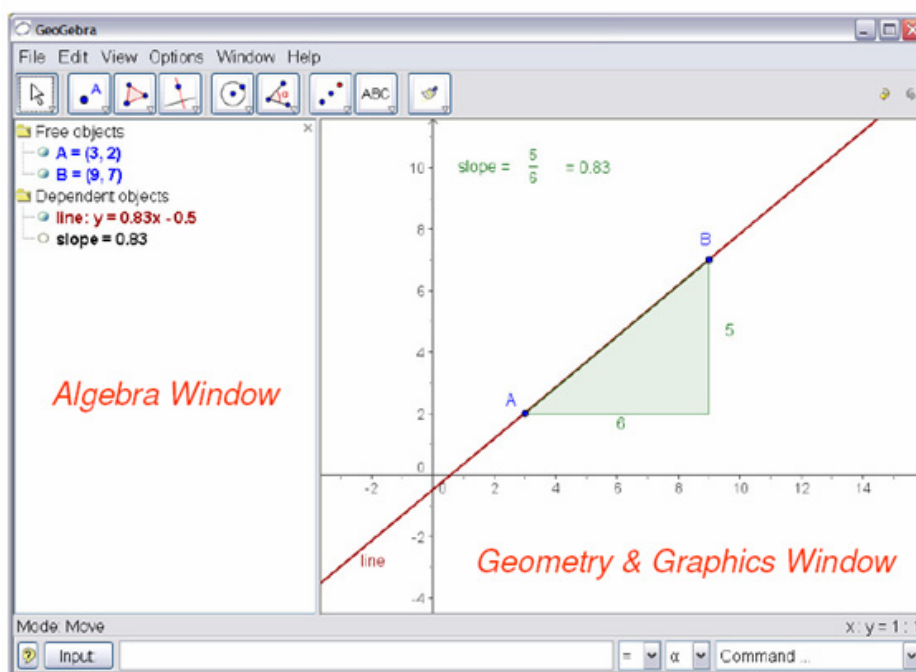

Figure 1: default *GeoGebra* screen showing both the algebra and geometry windows

As such, *GeoGebra* provides the facility to move between the algebra window and the geometry window - meaning that it is possible for the user, on the one hand, to investigate, say, the parameters of the equation of a circle by dragging the circle with the mouse and observing the equation change, or, on the other hand, the user can change the equation of the circle directly and observe the way the objects in the geometry window change.

Figure 2 illustrates the *GeoGebra* screen for a problem involving the tangent to an ellipse (with the language set to German). Again, the user can drag objects in the geometry window and observe the changes in the algebraic representation; or the user

can change the algebraic representations in the algebra window and observe the objects change in the geometry window.
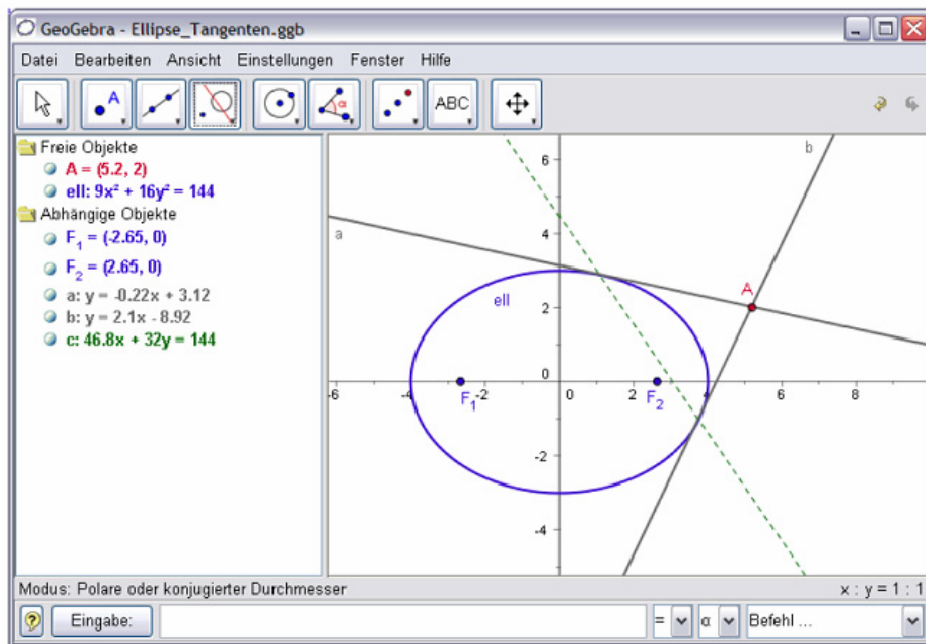

Figure 2: *GeoGebra* screen showing a problem involving the tangent to an ellipse

The software has a range of other features that make it useful for tackling a range of problems in mathematics. For example, in Figure 3 "sliders" are employed to provide a way of dynamically adjusting parameters.
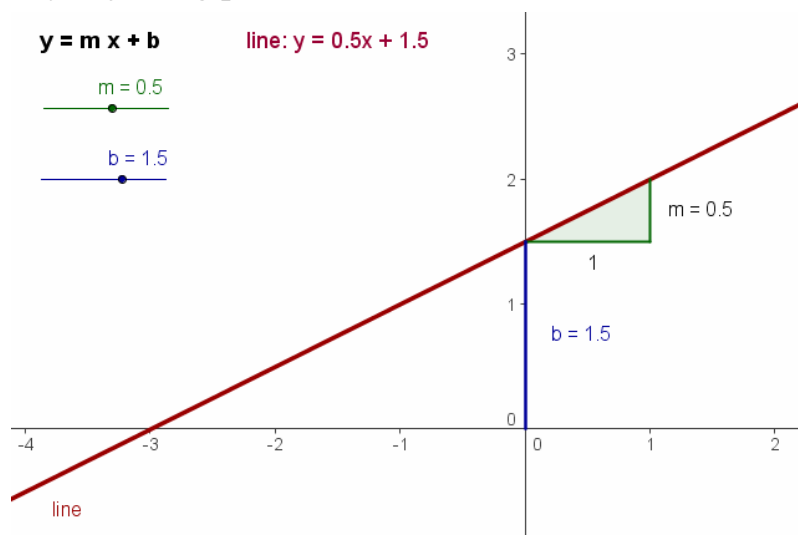

Figure 3: using sliders in *GeoGebra* to visualise parameters

Thus the basic objects in *GeoGebra* are not only points, vectors, segments, polygons, straight lines, all conic sections, etc., but also functions in x. Hence it is possible to enter directly the coordinates of points or vectors, the equations of lines or conic sections, plus functions, numbers, angles, and so on. For example a line g may be entered as g: $3x + 4y = 7$ or a circle c as c: $(x - 3)^2 + (y + 2)^2 = 25$.

This capability means that calculations with geometric objects like points and vectors are feasible. For example, the centroid of a triangle with vertices A, B and C might be entered as $S = (A + B + C) / 3$. Additionally, *GeoGebra* offers many powerful commands starting from the slope of a straight line and encompassing differentiation and integration of functions.

On top of all this, *GeoGebra* is truly platform-independent (in that it is written in *Java*) and it is free to use directly from the *GeoGebra* website (see endnote). Not only that, but any *GeoGebra* construction can be exported/saved as a webpage (in html format) and hence can become a "dynamic worksheet" for learners to use. *GeoGebra* is also freely translatable into other languages (not only its menus but also its commands) and is currently available in upwards of 35 languages.

Given the capabilities of *GeoGebra* and its potential in mathematics education, it is worth considering the relationship between geometry and algebra in the school curriculum.

GEOMETRY AND ALGEBRA IN THE SCHOOL CURRICULUM

While, as noted above, geometry and algebra are afforded prominent positions in school curricula around the world, the way the topics are treated, and the attention paid to inter-relationships between them, can vary quite significantly. For example, in some countries there are entirely separate courses on geometry and algebra, while in other countries, the curriculum alternates topics from the various components of mathematics. Not only that, but the relative emphases given to geometry and algebra in school mathematics curricula is also subject to change, with it likely that the emphasis given to geometry has been reducing relative to algebra over the past fifty years (Jones, 2000; 2002; Jones & Mooney, 2003)

In some ways these changing relative emphases given to geometry and algebra in school mathematics reflect what has been, in the history of mathematics, a somewhat (and sometimes) uneasy relationship between these topics (Atiyah, 2001; Giaquinto, 2007; Kvasz, 2005). According to Atiyah (2001), fundamental to what can seem like a dichotomy is that:

> "Algebra is concerned with manipulation in time, and geometry is concerned with space. These are two orthogonal, aspects of the world, and they represent two different points of view in mathematics. Thus the argument or dialogue between mathematicians in the past about the relative importance of geometry and algebra represents something very, very fundamental."

While algebra provides powerful techniques, Atiyah sees a danger that "when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning". Hence, as Giaquinto (2007) argues, presenting the relationship between geometry and algebra as a dichotomy is something of an over-simplification, arguing that "the algebraic-geometric contrast, so far from being a dichotomy, represents something more like a spectrum" (p240).

A recognition of the somewhat false dichotomy between geometry and algebra has led to articles in support of something more like mutual inter-dependence (for example, Davis, 1998; Diemente, 2000; – though see Farrell & Ranucci, 1973), including highlighting the role that technology could play (see, for example, Erbas, Ledford, Orrill & Polly, 2005). A single piece of software, like *GeoGebra*, that combines geometry and algebra, could be a powerful tool in mathematics education, but to have an impact there are implications for the pre-service education, and inservice professional development, of teachers of mathematics.

IMPLICATIONS FOR TEACHER DEVELOPMENT

As Hohenwarter and Lavicza (2007) explain, for the majority of teachers, solely providing technology is insufficient if technology such as *GeoGebra* is going to be fully integrated into their teaching. To date, for teachers using *GeoGebra*, online collaboration is being supported via a *User Forum* (whereby users can help other users) and a *GeoGebraWiki* (a way of pooling and sharing teaching materials). A beneficial next step would to be able to offer professional development for teachers and coordinate research activities in relation to *GeoGebra*. In terms of working with pre-service mathematics teachers, some ideas and examples are provided by Kokol-Voljc (2007) and by Edwards and Jones (2006).

CONCLUDING COMMENTS

As a freely-available open-source software package that combines both geometry and algebra, *GeoGebra* has much to offer. It is perhaps worth concluding with the words of Edwards and Jones (2006) that utilising software like GeoGebra could inspire a change to forms of classroom problems "that need high-level thinking, and things that students may find themselves wanting to follow-up outside of regular school lessons" (p30).

NOTES

GeoGebra: www.geogebra.org

GeoGebraWiki: www.geogebra.org/wiki

GeoGebraForum: www.geogebra.org/forum

International GeoGebra Institute: www.geogebra.org/IGI

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The BSRLM geometry working group focuses on the teaching and learning of geometrical ideas in its widest sense. Suggestions of topics for discussion are always welcome. The group is open to all.