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ARITHMETICAL NOTATING AS A DIAGRAMMATIC ACTIVITY

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Qualitative data is presented from the trialling of a software-based arithmetical notating task designed to foster engagement with the structure of equality statements. The design rationale is "diagrammatic activity" (Dörfler, 2006) where arithmetic inscriptions onscreen are observed and manipulated according to operational rules. The data suggest that the children's readings of arithmetical notation were transformed from computation to pattern awareness and substitution making. This afforded the emergence of commutative and partitional meaning making for $a+b=b+a$ *and* $c=a+b$ *syntaxes respectively.*

It is widely reported in the literature that children attend more readily to computational than structural readings of arithmetical equivalence statements (Behr et al., 1976; Jones, 2006; Kieran, 1981; Molina, 2006). These computational readings are deeply entrenched and can impact negatively when formal algebraic notation is encountered at the start of secondary schooling (McNeil & Alibali, 2005). In this paper I report on a trial of an arithmetical notating task designed to foster meaningful engagement with the partitional and commutative properties of equality statements.

TASK DESIGN

The design rationale for the task is Dörfler's (2006) interpretation of Charles Sanders Peirce's "diagrammatic reasoning" as applied to mathematics learning. According to Dörfler's view, the numerals, operator signs and other inscriptions that comprise arithmetic and algebra can be considered as the objects of mathematical activity, rather than as representational referents to abstract mathematical objects. An inscription can be considered a "diagram", in the Peircean sense, when it can be observed and manipulated according operation rules. For example, the inscription $30+41$ can be transformed into $70+1$ by substituting 41 with $40+1$ and substituting $30+40$ with 70. The power of carefully designed diagrammatic tasks is that learning mathematics becomes an empirical activity involving making discoveries through experimentation with physical inscriptions on a page or computer screen.

The remainder of this section describes a computer program and task designed to afford learners diagrammatic engagement with arithmetical notation[1]. The software, called *Sum Puzzles*, was set up to present a sequence of 11 puzzles. Each puzzle presented a term, or "sum" lacking an equals sign, such as $8+7+1$, and quality statements that can be viewed as tools for acting on the sum (see Figure 1).

The software supports two basic functionalities: selecting equality statements and substituting terms. A statement is selected by clicking on the equals sign which causes the statement to be highlighted ($8+7=15$ has been selected in Figure 1). In this sense

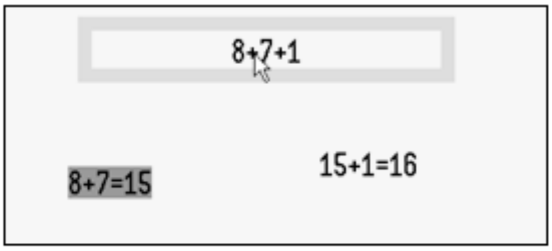

Figure 1

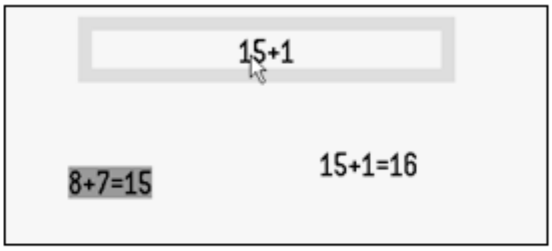

Figure 2

the inscription = might be thought to act as a handle for taking hold of equality statements. A term is substituted by clicking on it, and the substitution (if any) is determined by the currently highlighted equality statement. For example, clicking the inscription $8+7$ in the sum in Figure 1 transforms it into 15, as shown in Figure 2. Substitutions are reversible. If the inscription 15 in the sum in Figure 2 were to be clicked again whilst the statement $8+7=15$ is still selected the situation would return to that shown in Figure 1. Substitutions can also be made on terms within equality statements. Given the highlighted statement $8+7=15$, if the inscription 15 in the statement $15+1=16$ were clicked it would become $8+7+1=16$. The observational process of finding like terms in order to make substitutions ("iconic matching") can be considered as a potentialised diagrammatic activity embedded within the software. It was conjectured that the dual activities of selecting statements and clicking terms would transform children's reading of notation from that of computation to noticing iconic matches and experimenting with substitutions. This in turn, it was conjectured, would afford them opportunities to construct commutative and partitional readings of $a+b=b+a$ and $c=a+b$ statements respectively.

METHODS

The trial reported here involved a pair of Year 5 boys, T and A, deemed mathematically able by their class teacher, working together with the software and task for a period of about 40 minutes. My role as researcher was to show children how to operate the software and to set challenges. Other than this I occasionally prompted for verbal elaborations (as in "Why do you think that didn't work?") and offered encouragement and praise. Data was captured as screen-movies of the children's interactions along with an audio track of their discussion. This data was transcribed using the qualitative analysis software package *Transana* and evidence for non-computational readings of equality statements sought.

DATA AND DISCUSSION

Engagement was high during the trial, as illustrated by the boys' responses afterwards:

1. A: It's quite challenging [T: Yeah] It's kind of addictive in a way. You don't want to stop. [T: Yeah] Kind of like, really determined.
2. T: Bit fiddly. And it can sometimes get really annoying but it's still fun.

During the first two puzzles, which contain only compositional statements, the boys' readings were computational, as would be expected from a reading of the literature:

3. T: I think it might be this one. Yeah, [A: Yeah] 'cause 7 add 1 equals 8, 8 add 8 is 16. There... then if you click on... click on the equals...
4. A: 7 add 1 is...
5. T: Yeah. Then, like
6. A: Yeah, 'cause 7 plus 7 is 14, [T: And then...] isn't it?

Early evidence for a shift towards iconic readings occurred during Puzzle 3 (see figure 3), which contains the first commutative statement. T highlighted $1+9=10$ and attempted to make a substitution on the term $9+1$ (R is researcher):

7. R: Why do you think that wasn't working?
8. T: [highlights $1+9=9+1$] Maybe because... 1 and 9 is...
9. A: Oh, because it hasn't got that sum in it.
10. R: What do you mean?
11. A: Well 'cause that's [$1+9=10$] got 1 add 9 but then the end of that's [$1+9=9+1$] got 9 add 1.
12. R: Okay, I see.
13. T: [clicks on the sum] There. 1 add 9.
14. A: Now try that one T, and...
15. T: Yeah.
16. A: Yes!

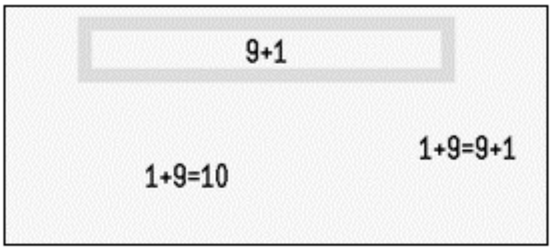

Figure 3

Following this both T and A came to use the verb "swap" when discussing and highlighting commutative statements throughout the trial. When they came to Puzzle 6 (Figure 4), which contains the first partitional statement, the boys initially used a computational reading to correctly eliminate the final statement ($70+1=71$) in the puzzle solving sequence from their choice of three statements (lines 17 to 24). Then T had an insight: despite not having encountered or used a partitional statement previously in the software, he inferred a "splitting" reading of $41=40+1$ (lines 25 to 30).

17. T: Er... 30 add 41. 30 add 40 equals 70.
18. A: Nah, maybe it's that one. Nah, it's that one. I think it's, maybe... that one looks like the closest to it. That one goes with that one, because that one's the sum you do at the end isn't it?
19. R: Which one's the sum you do at the end?
20. T: That one [indicates $70+1=71$].
21. A: Because 70 plus 1 because you get sevent- that must be the same.

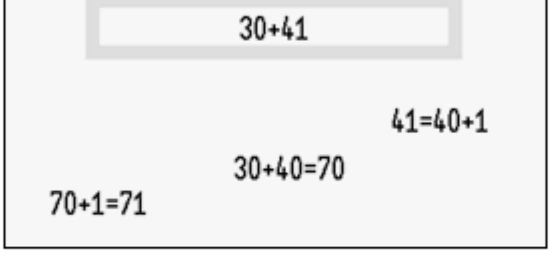

Figure 4

22. R: How do you know that's the one you do at the end?
23. A: Well it's already got half the answer. Because 30 add 40...
24. R: Oh I see, yeah.
25. T: Oh! That's [$41 = 40 + 1$] the one that you do first! That has to be.
26. R: Why?
27. T: Because it's splitting up the 40 and the 1.
28. R: Ok.
29. T: So maybe if you... click on there now. Click on 41. Try that. Yeah.
30. A: Yes.

Over the duration of the trial the boys came to use the verbs “swap” and “split” when using commutative and partitional statements and developed an effective strategy for making sense of increasingly complicated puzzles. This involved identifying partitional statements in order to break sums up into constituent parts, then swapping and composing these parts into a final number. The following excerpt demonstrates the power of this approach, when T efficiently solved Puzzle 10 (Figure 5), which contains eight statements, and is significantly more complicated than any of Puzzles 1 to 9, which contain a maximum of five statements. Note, however, that despite an emerging grasp of the diagrammatic operation rules, T still repeatedly attempted to make iconically mismatched substitutions using $5 + 7 = 7 + 5$.

31. T: Okay! 65 add 87, any spli... 8... Yeah. That one. That splits it up. [highlights $87 = 80 + 7$ and clicks 87 in $65 + 87$; highlights $65 = 5 + 60$ and clicks 65 in $65 + 80 + 7$] Erm... 50... 5 add 60... 80
32. A: Yeah.
33. T: [highlights $80 + 60 = 140$ and clicks centre of $5 + 60 + 80 + 7$] No. Oh you need to move 'em, you need to swap 'em round.
34. A: Hm.
35. T: [highlights $7 + 5 = 12$ and clicks 5 then 7 in $5 + 60 + 80 + 7$] No. Those need to be swapped round as well. ... There. [highlights $5 + 7 = 7 + 5$ and clicks 5 then 7 in $5 + 60 + 80 + 7$] Swap round!
36. R: Why wouldn't it swap them round?
37. T: Because there's something in between?
38. R: Okay.
39. T: There that one. [highlights $80 + 60 = 60 + 80$ and clicks centre of $5 + 60 + 80 + 7$] There. Now you can do that one. [highlights $80 + 60 = 140$ and clicks centre of $5 + 80 + 60 + 7$] 140. Hopefully that will do it now. [highlights $5 + 7 = 7 + 5$ and clicks 5 then 7 in $5 + 140 + 7$] No. [pause] Ah! [highlights $140 + 5 = 5 + 140$ and clicks 140 in $5 + 140 + 7$] That swapped it round. ... There. [transforms $140 + 5 + 7$ into 152 using $5 + 7 = 7 + 5$, then $7 + 5 = 12$, then $140 + 12 = 152$] And... Yes!

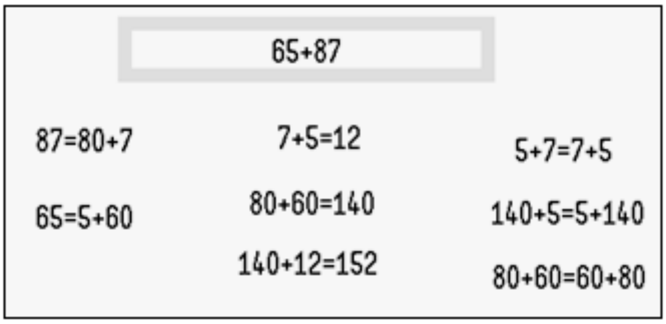

Figure 5

Following this success, I reset Puzzle 10 and posed an alternative challenge: modify the puzzle such that the sum $65+87$ can be changed into its answer in a single click. In the case of Puzzle 10, this involves transforming the statement $140+12=152$ into $65+87=152$, and then using it to substitute the sum for 152 in a single click (the boys were not told this explicitly). This subtle change in task goal, without alteration to the diagram or its underlying operational rules, impacted significantly on the boys puzzle solving strategies (compare the T's confident competence in lines 31 to 39 with the uncertainty in lines 40 to 52). Initially A was drawn to compositional statements (line 40). When T initiated their emerging and successful strategy of starting with highlighting a partitional statement (line 43) they appeared unsure of where exactly to make their substitution, and tried iconic mismatches:

40. A: So, you usually use these things [indicates $7+5=12$].
41. T: Well that's [indicates $87=80+7$] what we started with wasn't it?
42. A: Hm.
43. T: So, erm, select that maybe. [A highlights $87=80+7$] Now click on...
44. A: Maybe 60 or... [indicates $65=5+60$]
45. T: Does 87 use... 8... 80.
46. A: [clicks 80 in $80+60=140$] No it doesn't do anything. [highlights $80+60=140$]
47. T: Erm, try that one.
48. A: [clicks 80 in $87=80+7$] Nope.
49. A: Which one's... [highlights $5+7=7+5$]
50. T: Erm, try and click... like that one. Now try 7 add 5 or something.
51. A: [clicks + in $7+5=12$] Yes.
52. T: Yeah, 5 add 7!

The boys continued in a similar vein for several minutes, taking a total of 8 minutes to complete this challenge, in contrast to the original challenge immediately prior to this which had taken them only 1 minute for the same puzzle. At times of impasse and frustration, they abandoned iconic matching entirely in favour of blind experimentation, though not with much expectation:

53. A: Try 12 equals 7 plus 5.
54. T: Maybe you have to do this or something. [clicks every numeral in $140+5=5+140$ and $12=7+5$]
55. A: Try all the numbers.
56. T: [laughs] Yeah, that's what I'm doing. [clicks every numeral in every other statement]
57. A: They're not going to work though.
58. T: Yeah, don't think it will.

Nonetheless, they made gradual progress in a two-steps-forward-one-step-back manner whenever they returned to iconic matching, until they were close enough to see the rest of the way and solve the puzzle.

CONCLUSION AND FURTHER WORK

The data demonstrate a shift from computation to the diagrammatic activities of iconic matching and substitution making over the duration of the trial. This in turn afforded commutative and partitional readings of $a + b = b + a$ and $c = a + b$ statements respectively. The boys developed sustained competence with diagrammatic activity, although still frequently attempted to substitute iconic mismatches. A slight alteration of the task goal impacted dramatically on the boys' emergent (and successful) puzzle solving strategies and resulted in iconic matching being entirely abandoned at times.

The data reported here is from the first trial of the software and task after its development. Seven more trials with seven different pairs of Year 5 children of varied mathematical ability (as deemed by their class teachers) have since been carried out, with minor modifications to the puzzles and research focus each time. Two of these trials were abandoned after about fifteen minutes because the children were not comfortable talking to one another and no verbal data was being generated. In the remaining five trials similar results were obtained to those reported here, notably the emergence of diagrammatic activity in place of computation over the duration of the trials. Emergent commutative readings are clearly evident across all five trials, and emergent partitional readings evident to a greater or lesser degree across three of the trials.

NOTES

1. The software and task was developed by the author in *Imagine Logo* (Kalas & Blaho, 2003).

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