USING DYNAMIC GEOMETRY TO INTRODUCE CALCULUS CONCEPTS: CALGEO AND THE CASE OF DERIVATIVE

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CalGeo is a three-year project supported by EU programme Comenius 2.1. Amongst the objectives of this project is the design of an in-service teacher education programme which employs dynamic geometry tools for teaching Calculus in upper secondary education. In this paper we present the project, its main objectives and the produced material; an example of a learning environment/activity designed for the introduction to the notion of derivative at Year 12; and, some results of the application of this activity in a real classroom situation. In this activity we use the tangent line and the property of local straightness to introduce the formal definition of derivative. Several cases of differentiable and non-differentiable functions are discussed through their geometrical and symbolic representations.

INTRODUCTION

Calculus has a wide field of applications in other disciplines and constitutes a basic part of the mathematical curriculum of secondary education. Calculus knowledge is also necessary for the successful study of several subjects at university. Nevertheless research shows that the majority of students face serious problems in understanding basic Calculus concepts (for example see Harel, Selden & Selden, 2006).

The work presented in this paper originates in a three-year project called *CalGeo* (Teaching Calculus Using Dynamic Geometric Tools). The main project objectives are:

i) the investigation and planning of a programme for secondary education which employs dynamic geometrical tools for the teaching of Calculus, and

ii) the design of in-service training course for mathematics teachers based on the above programme.

The project focuses on the following topics: introduction to infinite processes, limit, continuity, derivative and integral. For each topic the training material includes documentation that raises mathematical, historical and didactical / pedagogical issues as well as a set of proposed activities. The produced material was tested in pilot teachers' training course as well as in real classroom situations in each of the participating countries.

The participants of this project are University of Athens (Greece), which is the coordinating institution; University of Crete (Greece); University of Southampton (United Kingdom); University of Cyprus; and, University of Sofia (Bulgaria).

In what follows we describe the rationale of the activities; the dynamic environment within which the activities were developed; and, we exemplify with an activity concerning the concept of derivative, as well some results from its implication in real classroom conditions.

LEARNING ENVIRONMENT / ACTIVITIES

The activities of the project were designed in order to be used towards the introduction of Calculus concepts at upper secondary education level (Year 11 and 12). They offer problem solving situations in which previous knowledge will turn out inadequate and the opportunity to explore alternative and generalisable aspects of an already known concept (e.g. the tangent line of the circle as the limiting position of secant lines).

The learning environments were designed in order to approach intuitively the corresponding mathematical notion(s) in ways that are consistent with formal mathematical theory (e.g. visual representation of the ε-δ definition of the limit) taking into account the students' previous knowledge and the topics which have proved to be a source of learning difficulties in calculus courses.

In this project we employed more than one dynamic geometry software (DGS). In Greece, we used a DGS called EucliDraw v.2.2.2. In addition to DGS facilities, this software offers a function editor / sketch environment as well as some tools appropriate for Calculus instruction. Indicatively, we refer to the 'magnification tool' that can magnify a specific region of any point on the screen in a separate window. This magnification can be repeated as many times as the user specifies through a magnification factor. Other useful, for Calculus, facilities are these that can partition an interval; construct the lower and upper rectangles covering the area defined by a graph and the $x'x$ axis; control the number of the decimal numbers of calculations etc.

For more information about the project, its theoretical assumptions, the dynamic environment and the produced activities see (Biza, Diakoumopoulos and Souyoul 2007) and in the project website: www.math.uoa.gr/calgeo.

ACTIVITY ON THE CONCEPT OF DERIVATIVE

The aims of this activity are: the introduction to the definition of the derivative at a point; the introduction to the definition of the tangent line of a function graph as the limiting position of the secant lines as well as the linear approximation of the curve at this point; the reconstruction of students' previous knowledge about tangent line grounded to the Euclidean Geometry context in order to be applicable in general cases of curves; the connection of the symbolic and geometric representations of derivative at a point; and, the recognition by the students the property of the "smoothness" of a function curve at a point and its relationship to the differentiability of the function at this point.

According to Tall (2003), the *cognitive root* of the notion of derivative is the *local straightness*. The property of *local straightness* refers to the fact that, if we focus close enough on a point of a function curve (a point at which the function is differentiable) then this curve looks like a straight line. Actually, this ‘straight line’ is the tangent line of the curve at this point. This property is valid in all cases of tangent lines and its understanding could be facilitated by the use of new technology with appropriately designed software (Tall, 2003; Giraldo & Calvalho, 2006). On the other hand the early experiences of the circle tangent contribute to the creation of a *generic tangent* as a line that touches the graph at one point only and does not cross it (Vinner 1991). Furthermore, students perceive not generally valid properties related to the number of common points or the relative position of the tangent line and the graph as defining conditions for a tangent line. Different combinations of these properties create intermediate models of a tangent line. This occurs through the assimilation of new information about graph tangents in the existing knowledge about circle tangent (Biza, 2007; Biza, Christou & Zachariades, 2006).

The activity starts with the notion of circle tangent in the context of Euclidean Geometry. The students are asked to sketch in the EucliDraw environment a circle; a point of it *A*; and, a line vertical to the radius *OA*. The tasks of the worksheet intend to make the students observe that the tangent line is the limiting position of the secant lines *AB* as *B* approaches *A* and with the help of the ‘magnification tool’ to magnify the region around *A* and observe that the circle looks like its tangent as the magnification factor increases.

In the next step – through the investigation of the tangent line in the case of the semicircle as a function graph – students make the transition to the Calculus context. Thereafter, the students work in an already constructed environment of EucliDraw and they are introduced to the tangent line of function graph and through this to the definition of the derivative. In this environment the graph of $f(x) = \sin(x)$ and a point $A(x_0, f(x_0))$ of it are sketched. In the display of this environment in Figure 1 we can notice some other constructions as: the number h; the points $B(x_0+h, f(x_0+h))$ and $C(x_0-h, f(x_0-h))$; the magnification window of a region of *A* related to a magnification factor equal to $1/h$; the secant lines *AB* and *AC*; and, the slopes of these lines.

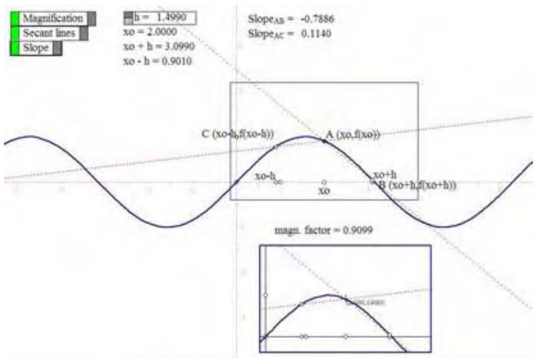

Figure 1: A EucliDraw environment for the introduction to the notion of

Students, following the tasks of the worksheet: decrease the values of h and through this increase the magnification factor and move the points *B* and *C* closer to *A*; observe what happens with the secant lines and their slope and express their slope symbolically.

Several cases of differentiable and non-differentiable functions are discussed through their geometrical and symbolic representations. For example: $f(x) = |\sin(x)|$ that is not differentiable at the points in which the graph intersects the x´x axis; $f(x) = x^3$ in which the $O(0,0)$ is an ‘inflection point’; and the $f(x) = \sqrt{|x|}$ in which the $O(0,0)$ is a cusp point.

APPLICATION OF THE LEARNING ENVIRONMENT

We applied the teaching material described in the previous paragraph in a Greek Year 12 classroom of 17 students (8 girls). The mathematics teacher and the first author conducted the application. The teacher was familiar with different types of educational software and their applications. In addition, he had followed the training programme of the Calgeo project.

All students had mathematics as a major subject (they were candidates for science or polytechnic studies in the university admission examination that, in Greece, takes place at the end of that year) but had varying levels of performance. By the time the application took place (at the end of the first semester), the students had been taught functions, limit, continuity and they were just before the introduction of derivative. Most of these students had previous experience of DG environments in their Euclidean Geometry lessons but not in Calculus lessons.

This application lasted two sessions (one hour and two hours, respectively) and took place in the mathematics classroom replacing the traditional lesson. The students had been split in groups of three or four and each group used one of five portable PCs whereas the instructors used a sixth one plugged into an LCD projector.

The teaching material consisted of the electronic environment designed in EucliDraw software and the students' worksheets. The records collected across this application included: pre and post questionnaires; audiotapes of the lessons; students' work in their worksheets; and, classroom transcripts.

Observation on the classroom application

In the first session dedicated to the application students became familiar with the novel instructional situation: the transformation of their classroom into a laboratory; the electronic environment; and the rules of collaboration and communication in the group and between the groups and the instructors. It was easy for those students to feel comfortable in the EucliDraw environment but very difficult for them to exchange ideas and work in the new classroom situation. The collaboration was established in the second session when the transition to Calculus context was beginning to be discussed. At that point and onwards several issues of discussion emerged.

Although sometimes the questions posed by the students appeared to be outside the main aims of the task, we tried to take advantage of these questions in ways that would help carry on with the activity. For example, questions like: "is the circle a function graph?" "what is the formula of the graph of the semi-circle?" proved very useful in terms of the transition from Euclidean Geometry to Calculus.

Conversation upon the image in the magnification window brought up questions like: "can we draw a tangent line at a vertex of a polygon?" or "how does the vertex look in the magnification window?" To clarify this, we made a parallelogram in EucliDraw and a line passing through a vertex of it. We magnified the region around this vertex and we discussed the figure in the magnification window (the image did not like a straight line at any region of the vertex). We could say that the representation of the *local straightness* proved very illustrative when the curve is not *smooth* and thus does not look like a straight line in the magnification window (e.g. when the point is vertex or when the function is not differentiable).

Some students following the instructions of their worksheet tried smaller and smaller values of *h* but not with smaller absolute value. This actually was a wrong statement in the worksheet and we grasped the opportunity to discuss the meaning of "the *h* tends to 0"

In the discussion about how can we define the tangent line in the case of function graphs, students used arguments based on both the context of Calculus and Euclidean Geometry context. For example, in the case of the $f(x) = \sin(x)$ graph and its tangent line at point $A(x_0, f(x_0))$ a student declared:

[S1]: The tangent line is a line that has one common point with the graph.

Then we moved the point *A* so that the constructed line (the limiting position of *AB* and *AC*) to cut the curve in another point:

[I]: What do you say now?

[S2]: We could say that the tangent line is a line that has one common point at a neighbourhood of the tangency point and does not intersect the curve at this point.

We kept this statement written in the whiteboard and later on the discussion of the $f(x) = x^3$ and the tangent at the point $O(0,0)$ the same students said:

[S2]: It looks like it is [a tangent] but it cannot

[I]: Why?

[S2]: It cannot be a tangent because it cuts the graph … how can I say that … it intersects … it splits it in two parts, on part on the one side and one on the other [of the line] … on the other hand it looks to coincide with the curve near point *A* at the magnification [window]… I don't know …

Through the discussion on several cases of graphs we tried to facilitate students' reconstruction of their previous knowledge about tangents. As the analysis of the post-application questionnaires revealed some students did make this reconstruction.

CLOSING COMMENT

The application of this specific material raised some issues concerning the teaching material and its application. Some of these issues concerned the student's adaptation in the new for them classroom environment (e.g. collaboration, communication, familiarity with electronic environments etc). Some others were related to this specific instructional approach (e.g. different representations, students' misunderstanding and the role of the several examples, accuracy of measurements in electronic environments, the visual perception of local straightness etc.)

These observations were demonstrative of the diverse situations that these environments created in the classroom community. Although further investigation and more systematic research are needed for more valid results we could say that this application proved very helpful for the development and elaboration of the material of our project.

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