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IS THE TANGENT LINE TANGIBLE? STUDENTS' INTUITIVE IDEAS ABOUT TANGENT LINES

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The results presented in this paper are a part of a doctoral study nearing completion which focuses on students' intuitive thinking about tangent lines. The study explores the ways in which students who have studied tangents in school in different contexts (Geometry, Analytic Geometry and Calculus): (i) perceive properties that are not generally valid as defining conditions of the concept and (ii) create new, often not-valid, properties out of the fusion of information from across the different contexts. Data was collected through a questionnaire administered to 182 first year mathematics undergraduates in Greece at the very beginning of their studies. Here I exemplify from the data analysis in the course of which several models of the tangent line have emerged based on the above properties.

THEORETICAL ASSUMPTIONS

Students in secondary school or at university often revisit concepts that they have studied earlier and at a more elementary level. In this case their total cognitive structure associated with a concept (the *concept images* according to Tall and Vinner (1981)) that has developed during their early studies has to be modified (for example, generalised) in order to be applicable in a broader context. This generalisation is a non trivial step in students' knowledge acquisition which is not a straightforward accumulative process. According to Harel and Tall (1989) there are different kinds of generalisation such as *expansive generalisation*, in which the student's existing cognitive structure is extent, without requiring changes of current ideas and *reconstructive generalisation*, in which the existing cognitive structure is reconstructed in order to widen its range of applicability. In the *reconstructive generalisation* the old *concept image*s have to be radically changed so as to be applicable in a broader context. When students cannot achieve this generalisation various misconceptions can arise. Some of these are caused by the student's effort to assimilate new information in their existing knowledge, although these two are incompatible. In the study this paper draws on I examine the case of the tangent line from this theoretical point of view.

Research (Tall, 1987; Vinner, 1982, 1991) has revealed that previous knowledge about tangent line as the tangent of a circle contributes to the creation of a *concept image* of the tangent line as a line that meets the curve at one point and may not cross the curve at this point (*generic tangent*). This image of the *circle* tangent influences students' understanding of the more general concept of the *curve* tangent.

Sometimes students who have studied tangents in different contexts may perceive as defining conditions properties that are not generally valid (Winicki & Leikin, 2000).

Some of these properties have been created implicitly by the adjustment of a property useful in one context as if it is adequate in another one. In a previous phase of the doctoral study this paper draws on (Biza et al., 2006), a study conducted with 12^{th} graders in Greek high schools who have met the tangent line in different contexts (Euclidean Geometry, Analytic Geometry and Calculus) several *characteristics* emerged regarding the relationship between the line and the curve. The five most prominent of these are as follows:

X1: The line has one common point with the whole curve.

X2: The line has one common point at a neighbourhood of the tangency point. This implies that the line may have other common points with the curve in the following ways: other tangency points visible on the graph (*X2A*); other common points not visible on the graph (*X2B*); other common points (tangency or not) visible on the graph (*X2C*).

X3: The tangent line coincides with the curve. In this case the tangent line could have infinite number of common points at any neighbourhood of the tangency point.

X4: The tangent line at an *inflection point.* This aspect of the tangent line is crucial as the tangency at an *inflection point* entails that the line *cuts through* with the curve at this point and *splits* the curve at different semi-planes.

X5: The *smoothness* (at the *tangency* point) as an assumption for the existence of a tangent line.

Usually, information about one of the above *characteristics* is not enough to give us an image about the ways a student thinks about tangent lines. In the following, I present the study briefly as well as some results that illustrate how different combinations of student choices regarding these *characteristics* form different models of their conceptions of tangent lines.

METHODS

Data reported in this paper was collected from a questionnaire administered to 182 first year university students (97 female) from Mathematics Departments in Greek Universities. All participants had been taught about the tangent line in Euclidean Geometry, Analytic Geometry and elementary Calculus courses during the 10th, 11th and 12th grade, respectively, but not at university as the questionnaire was administered at the beginning of their tertiary education. I will present the part of the questionnaire (four questions) that relates to the above five characteristics (Fig. 1).

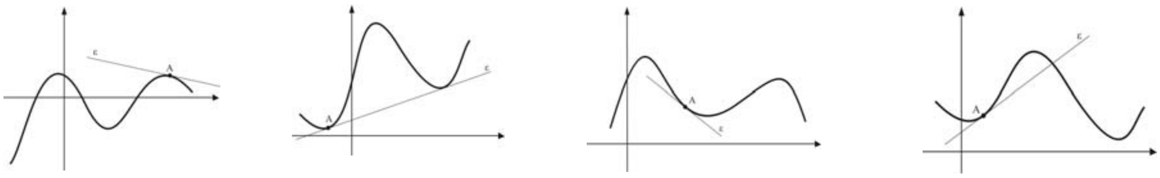
In Questions 1 and 2 the students are asked to write some informal ideas about the tangent line and its relationship with the corresponding curve. In these two questions there is not a unique correct response and there is no need for the students to answer by using formal / symbolic expressions. While completing the questionnaires the students were encouraged to write freely what they were thinking, sketch a figure, write in symbols or formulas, and generally use whatever informal or formal means to communicate their thoughts.

Questions 3 and 4 aimed to investigate students' ability to recognize and construct a tangent line of various curves in a graphic context, where no formula was given. Students were asked to justify their choices. There were 14 and 15 curves in the third and fourth Question, respectively but for the needs of this paper I will present only some relating to each of the five characteristics (as suggested in the *X* codes below each graph in Fig.1).

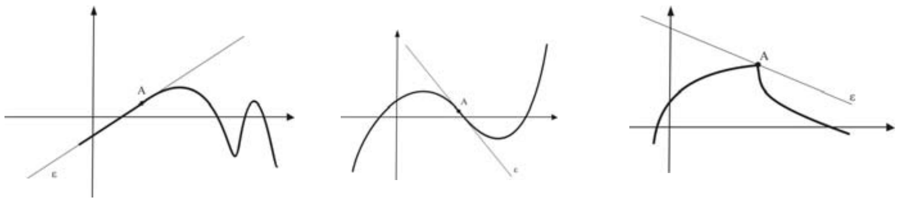
Question 1: Try to explain, in simple words, what you are thinking when you hear the term "tangent line"

Question 2: A line is the tangent of a curve at a point *A*. Write as many properties you can think about the relationship between this curve and its tangent line at point *A*.

Question 3 (Recognition): Which of the lines that are drawn in the following figures are tangent lines of the corresponding graphs at point A? Justify your answers.

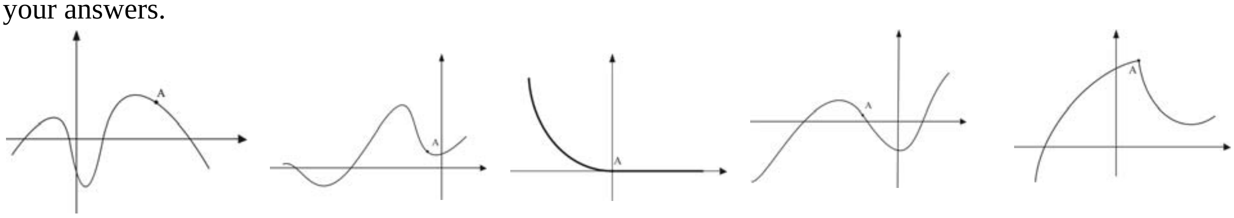


3.a (*X1*) 3.b (*X2A/C*) 3.c (*X2B*) 3.d (*X2C*)



3.e (*X3*) 3.f (*X4*) 3.g (*X5*)

Question 4 (Construction): Sketch the tangent lines of the following curves at point A, if they exist. Justify your answers.



4.a (*X1*) 4.b (*X2B/C*) 4.c (*X3*) 4.d (*X4*) 4.e (*X5*)

Figure 1: Questionnaire

Student choices in Questions 3 and 4 were characterised regarding their *correctness* and analysed quantitatively (Biza & Zachariades, 2006b). Student responses to Questions 1 and 2 as well as their choice/reasoning in Questions 3 and 4 were analysed qualitatively. As part of this analysis a coding was produced that described students' choices to the above *characteristics*, the context and the representations used in the responses and the consistency of student choices. Also for each student, a small narrative was produced describing the overall student profile as emerging from their response to the questionnaire (Miles & Huberman, 1994).

ANALYSIS

Through the analysis eight models of students' conceptions about the tangent line are revealed. These models are derived by different combination of choices regarding the five *characteristics*. All the students, except 14, are classified in these eight models. The students' responses to *t*he Tasks (Fig. 1) are described in Table 1: numbers in brackets indicate the number of students classified to the model and "✓" or "✗" indicates acceptance or rejection of the tangent line, respectively.

	3.a (*X1*)	3.b (*X2A*)	3.c, 4.b (*X2B*)	3.b/d, 4.b (*X2C*)	3.e, 4.c (*X3*)	3.f, 4.d (*X4*)	3.g, 4.e (*X5*)
***M1* (18)**	✓	✓	✓	✓	✓	✓	✗
***M2* (33)**	✓	✓	✓	✓	✗	✓	✗
***M3* (13)**	✓	✓	✓	✓	✓	✗	✗
***M4* (42)**	✓	✓	✓	✓	✗	✗	✗
***M5* (27)**	✓	✓	✗	✗	✓	✗	✗
***M6* (15)**	✓	✗	?	✗	✗	✗	?
***M7* (13)**	✓	✓	✗	✗	✓	✓	✗
***M8* (7)**	✓	✗	✗	✗	✗	✓	✓

Table 1: Responses to the Tasks

In the first model (*M1*) the tangent line could have more than one common point with the curve, there is a tangent line at the *inflection* points and there is no tangent line at the *edge* points. In this model the coincidence of line and curve is acceptable. In the second model (*M2*) the tangent line could have more than one common point with the curve under the condition that there is a neighbourhood of the tangency point where this point is the only common point. There is a tangent line at the *inflection* points and there is no tangent line at the *edge* points. In this model, contrary to the first one, the coincidence of line and curve is not acceptable. The third (*M3*) and the fourth (*M4*) models are similar to the first and the second, respectively, except the cases of the *inflection* points where the tangent line is not acceptable. In the fifth model (*M5*), the tangent line could have more than one common point with the curve under the condition that all of them are *tangency points*. Additionally, there is no tangent line at the inflection points and there is no tangent line at the edge points. In this model the tangent line could coincide with the curve as all the common points are *tangency points*. In the sixth model (*M6*) the tangent line has only one common point with the curve and there is no tangent line at the *inflection* points and probably there is no tangent line at the *edge* points. Obviously, according to this model the tangent line cannot coincide with the curve as it has more than one common point with the curve. However, sometimes students classified in this model accept or sketch a line, if the other common points are not visible/sketched (*X2B)*. In the seventh model (*M7*) the tangent line could have more than one common point with the curve under the condition that all of them are tangency points. There is a tangent line at the *inflection* points and there is no tangent line at the *edge* points. Finally, according to the eighth model (*M8*) the tangent line has only one common point with the curve. As a result the tangent line cannot coincide with the curve, can pass through an *inflection* point and probably through an *edge* point with the only restriction this point to be the only common point between the curve and the tangent line.

EXAMPLES OF MODELS *M4* AND *M6* (DIFFERENCES/SIMILARITIES)

In the above description of the models, *M3, M4* and *M7* describe students' local application of the properties described in models *M5, M6* and *M2*, respectively.

For example, according to the properties of the tangent line in *M4*, one *personal definition* (with the meaning given to the term by Tall and Vinner, 1981) of the tangent line could be: *The tangent line of a curve is a line that has (at least) one common point with the curve, (has the same slope with the curve at this point) and there is a neighbourhood of the tangency point in which the corresponding part of the curve remains at the same semi-plane of the line.* A student classified in this model writes in Question 1: "[…] all the points of the function [he means the function graph] are on the left or the right of the tangent line (at least the nearby points) […]". In Question 2: "If (ε): tangent and $f(x)$: function then $f(x)>(\varepsilon)$ or $f(x)<(\varepsilon)$ $\forall\, x \in [\alpha,\beta]$. It depends where the point A is". The same student accepts the line in Tasks 3.a, 3.b, 3.c, 3.d as tangent lines by writing in the last one "It has only one common point and $f(x)>(\varepsilon)$". But he rejects the lines in Tasks 3.e, 3.f and 3.g. In question 4 he sketches the *right* lines at 4.a and 4.b but he declares that 'there is no tangent line" in Tasks 4.c, 4.d, 4.e.

As another example, a *personal definition* of the tangent line for the students classified in the *M6* could be: *The tangent line of a curve is a line that has (exactly) one common point with the curve, (has the same slope with the curve at this point) and all the curve remains at the same semi-plane of the line.* A student classified in this model writes in Question 1: "The tangent line is the line that cuts the curve at only one point and itself or its extension doesn't split the curve in two semi-planes". The same student in Task 3.a, accepts the line by writing "… it has only one common point with the curve (the point A) and the whole curve is in the same semi-plane". Later, in Tasks 3.c and 3.d she rejects the line because "[…] it cuts the curve at another point. Thus it splits the curve into different semi-planes". In Task 3.f she writes: "The line is not a tangent because although it has only one common point with the curve, it splits it in different semi-planes" and in 3.g "although the line satisfies the presuppositions to be a tangent, it isn't because we don't have a curve"

CONCLUSIONS

Through the above spectrum of models of the tangent line, we can observe that the circle tangent properties influence student choices although they are not always directly used. Students have transformed these properties into new ones. The *M1* could be concerned as the most adequate as students classified in it seem to have been released from the constraints of the Euclidean Geometry context. Concerning the other models different kinds of mathematical (in)sufficiency in the student responses are revealed. Overall there is no need to discuss these models in hierarchical terms as they do not represent each a more (or less) correct mathematical behaviour. What is significant about the models is that they reveal the different ways in which students apply, or reconstruct and apply, their previous knowledge in new more general contexts as well as the ways in which this reconstruction takes place in a more (or less) effective fashion.

One of the main differences between the Euclidean and the Analytic point of view is that in the first the focus is global whereas in the second the focus is local. In the global point of view there are holistic properties about the relationship of the line with the whole curve. Students, in their effort to meet the needs of the general definition of the tangent line and *make the tangent line more tangible* in the Calculus context, adapt the previously known global properties to a different context where a more local consideration of properties is necessary. The detailed examination of this adaptation carried out in the context of this study aims to clarify students' conceptions of the tangent line and offer rich information that could contribute towards the construction of appropriate teaching environments that facilitate the students' transition from a geometrical to a Calculus context.

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