Hewitt, D. (2006) Proceedings of the British Society for Research into Learning Mathematics 26(3) November 2006

OBSERVATIONS ON THE DEVELOPMENT OF STRUCTURAL REASONING IN A FOUR-PHASE TEACHING SEQUENCE

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We examine the written responses of fifteen students (aged about 14½ years) to a homework task and their responses to the same task in a subsequent lesson. Students were asked to make observations about the sum of three consecutive numbers and to explain why they thought their observations were true, thereby giving students the opportunity to engage in structural reasoning. The teaching sequence had four phases designed to allow students to make, share and develop their observations and reasoning, and we found a clear improvement in the quality of students' responses. As far as students' reasoning is concerned, this suggests limitations may stem at least in part from a lack of familiarity with the nature of mathematical reasoning.

INTRODUCTION: PROOF AND STRUCTURAL REASONING

Formal proofs and mathematical argument both require the ability to make inferences and deductions on the basis of mathematical properties and structures, that is to engage in 'structural reasoning'. This can be contrasted with reasoning based on perception, appeals to authority (such as the teacher or a text-book), or empirical examples (see for example Harel & Sowder, 1998).

There is a large body of research which indicates that school students tend to engage in 'empirical reasoning' rather than argue on the basis of mathematical structure (Bell, 1976; Balacheff, 1988; Coe and Ruthven, 1994; Bills and Rowland, 1999; Healy & Hoyles, 2000). A partial explanation for this may be that structural reasoning is cognitively more demanding than empirical reasoning. On the other hand, there is also evidence to suggest that if students are required to focus on properties and relationships, as in some carefully designed computational environments (Hoyles and Healy, 2000) then some will engage in structural reasoning. Unfortunately, however, the school curriculum tends to foster empirical reasoning, at least in the UK, and especially since the 1990s when coursework involving 'investigations' was included in the national mathematics examination at age 16. As a result of being examined, investigational work has become more procedural, with students being encouraged to generate systematic data, to record these in a table, and to look for patterns (see Morgan, 1997)[1].

Thus it is possible that some students who adopt empirical reasoning do so as much for socio-cultural as for cognitive reasons. In particular, some students may have "never learned what counts as a mathematical argument" (Dreyfus, 1999). We examine this possibility in this paper and provide evidence to suggest that presenting students with examples of mathematical arguments, even without stressing their significance, can have a positive effect on the quality of their responses.

CONTENT AND STRUCTURE OF THE TEACHING SEQUENCE

The work reported in this paper was part of the Proof Materials Project (PMP) in which we collaborated with groups of teachers to develop materials and classroom approaches to enhance school students' proof and reasoning skills. Here we report on written responses to a homework task that was given to a class of students in Year 10 (aged about $14^1/_2$ years), and further responses to the same task in a single, follow-up lesson. For homework, students were asked to attempt these two questions, printed on an A4 sheet of paper with spaces to write their responses:

1. **Prove that the sum of two consecutive numbers must be odd.**
2. **Take any three consecutive numbers and add them together.**
   a) **What do you notice about the totals?**
   b) **Try to explain why this always happens.**

Following the homework, the whole of the next lesson (about 45 minutes) was devoted to these two questions, though in this paper we only consider Question 2.

We deliberately chose to present Question 2 in an open form, rather than using a closed version such as "Explain why the sum of three consecutive numbers is (say) a multiple of 3". We felt that this would make the task more engaging and the search for explanations more purposeful, and that it would be be instructive for students to compare the different conjectures that might arise (Watson, 2006).

In consultation with the teacher, it was agreed that the work on the task would proceed in four phases (in the manner of Hershkowitz and Schwarz, 1995). This would give students the opportunity to develop their own ideas, to share these with their peers and hence justify, evaluate and develop them further, and, finally, to demonstrate how their ideas had changed. Thus we wanted to capture some of the characteristics of an 'inquiry mathematics' classroom (eg Cobb and Yackel, 1998). In addition, given the difficulties students can have in trying to work together in a mutually beneficial way (Sfard, 2001), and the benefits of revisiting a task (Voutsina and Jones, 2005), we deliberately allowed time for students to work on their own so that they could start to form their own ideas and later perhaps restructure them.

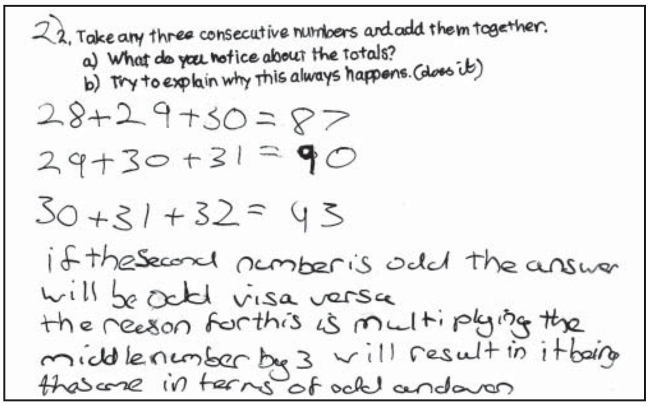
In **phase 1**, students worked on the task on their own for homework; in **phase 2**, at the beginning of the follow-up lesson, students discussed their responses to the task in groups (of up to 6 students), with each group being asked to produce an overhead transparency (OHT) that summarised their ideas; in **phase 3** students from each group displayed and explained the contents of their OHT to the rest of the class; finally in **phase 4**, towards the end of the lesson, students were asked to try the homework task once more, on their own.

The class teacher was experienced and highly skilled and the lesson flowed smoothly (perhaps also because the students did not want to show themselves up in the presence of a visitor). However, the teacher did not usually work with this class in this way, so that the skills and socio-mathematical norms (Cobb and Yackel, 1998) required for effectively generating, sharing and evaluating mathematical ideas were not particularly well-developed. Also, the teacher took a low key role, with the emphasis on managing the lesson rather than on developing or promoting particular mathematical ideas. This was done for two reasons: to emphasise that we were interested in and valued the students' own ideas, and to reduce the possibility that students would evaluate these ideas on the basis of the teacher's authority.

THE STUDY

The class was a top mathematics set[2] in Year 10 (aged about $14\frac{1}{2}$ years) of about 25 students, in a school situated in a relatively deprived area of a large, thriving town. We concentrate here on 15 students for whom we have completed phase 1 and 4 response-sheets and their group-OHTs. Our findings suggest that most of these students benefited from the work, in that they incorporated into their phase 4 responses some of the more powerful ideas encountered in phases 2 and 3.

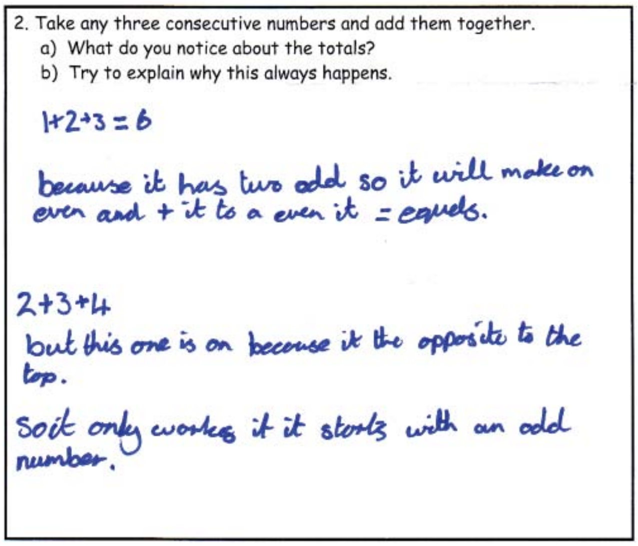
On Question 2 one might expect (or hope) that in part a) students will notice that the required sum is always a multiple of 3 (and indeed that it is 3 times the middle consecutive number). In the event, only 5 of the 15 students referred to this property in their phase 1 responses. In contrast to this, 9 students gave responses concerning parity in phase 1, such as this: 'if the first number is odd, the sum is even'. (This high frequency might have been partly due to the fact that Question 1 was explicitly concerned with parity.)


2) 2. Take any three consecutive numbers and add them together.
a) What do you notice about the totals?
b) Try to explain why this always happens. (does it)
28+29+30=87
29+30+31=90
30+31+32=93
if the second number is odd the answer will be odd visa versa
the reason for this is multiplying the middle number by 3 will result in it being the same in terms of odd and even

We managed to obtain the phase 2 OHTs of four groups. All the OHTs focussed on parity, as in this example (right) which came from a group of 5 boys. Here, the students have presented a systematic set of numerical examples, but have not made the common observation that in this case the sum increases by 3. Rather, they refer to the parity of the sum: "If the second number is odd the answer will be odd (and) visa [sic] versa". They have also given a justification for this, based on the fact that the sum is equivalent to 3 times the middle number (though this is not made fully explicit) and that multiplying a number by 3 does not change its parity. However, they have not explained why the sum is equal to three times the middle number.

All 5 students in this group gave stronger responses in phase 4 than in phase 1. Thus for example, one of the students, John, had focussed on parity in phase 1 (below, left), but switched to the "3 × the middle number" property in phase 4 (below, right). Moreover, he gave a very nice structural argument involving compensation to justify this property:

> If you take 1 from the 12 it makes 11, and add the 1 to the 10 and it makes it 11...



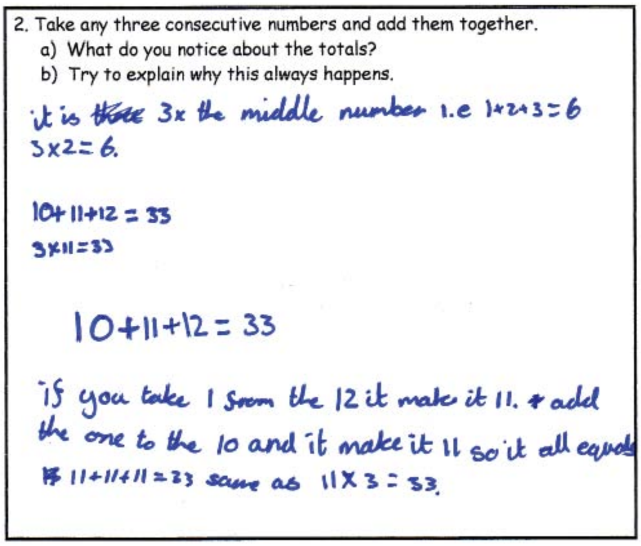
2. Take any three consecutive numbers and add them together.
a) What do you notice about the totals?
b) Try to explain why this always happens.

1+2+3 = 6

because it has two odd so it will make an even and + it to a even it = equels.

2+3+4
but this one is on because it the opposite to the top.

So it only works if it starts with an odd number.



2. Take any three consecutive numbers and add them together.
a) What do you notice about the totals?
b) Try to explain why this always happens.

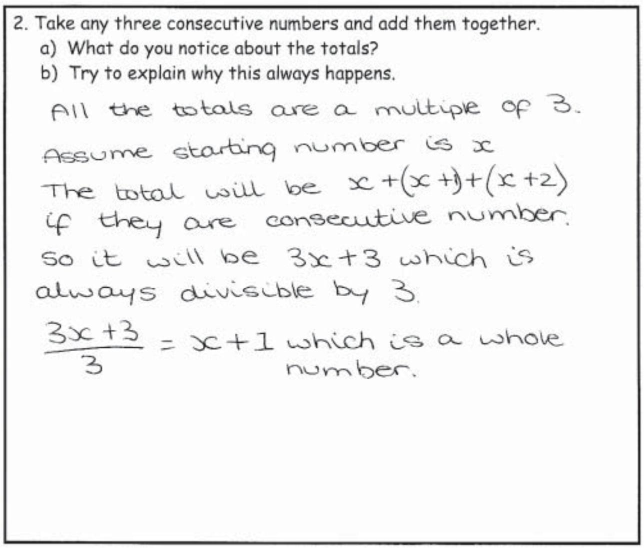
it is ~~three~~ 3x the middle number i.e 1+2+3=6
3x2=6.

10+11+12 = 33
3x11=33

10+11+12 = 33

If you take 1 from the 12 it make it 11. + add the one to the 10 and it make it 11 so it all equals
11+11+11=33 same as 11X3 = 33.

Only one of the 15 students, Avril, used algebra in their phase 1 response (which is shown below). She was not able to get her group to include this on their OHT (perhaps through reticence or perhaps because the others were not receptive). However, the teacher became aware of this response and in a rare intervention asked Avril to write it on a transparency which was then shown to the class.



2. Take any three consecutive numbers and add them together.
a) What do you notice about the totals?
b) Try to explain why this always happens.

All the totals are a multiple of 3.
Assume starting number is x
The total will be $x+(x+1)+(x+2)$
if they are consecutive number.
so it will be $3x+3$ which is always divisible by 3.
$\frac{3x+3}{3} = x+1$ which is a whole number.

This had quite a marked effect on the class, since, in addition to Avril, 5 of the 15 students included some algebra in their phase 4 response. It emerged that Avril had arrived at her phase 1 response in discussion with her father, and it is interesting to note that her phase 4 response lost some of the precision and technical language used in phase 1.

The five contingency tables, below, compare all 15 students' responses in phases 1 with those in phase 4, according to whether they include these features:

A. reference to the **parity** of the sum
B. reference to the sum being **3 times** the middle number (or a multiple of 3, or 3 times the first number plus 3)
C. a systematic set of sums that are observed to increase by a fixed **increment**
D. the use of a **compensation** argument to explain feature B or C
E. the use of **algebra** to represent the consecutive numbers and/or their sum.

The top-left cell (yellow) shows where students included a feature in their phase 4 response that they had not used in phase 1, while the bottom-right cell (blue) shows where they dropped a feature in phase 4 which they had used in phase 1.

parity

		phase 1 NO	phase 1 YES
phase 4 YES	8	3	5
phase 4 NO	7	3	4
		6	9

times 3

		phase 1 NO	phase 1 YES
phase 4 YES	12	8	4
phase 4 NO	3	2	1
		10	5

increment

		phase 1 NO	phase 1 YES
phase 4 YES	3	2	1
phase 4 NO	12	12	0
		14	1

compensation

		phase 1 NO	phase 1 YES
phase 4 YES	4	4	0
phase 4 NO	11	11	0
		15	0

algebra

		phase 1 NO	phase 1 YES
phase 4 YES	6	5	1
phase 4 NO	9	9	0
		14	1

As can be seen, generally the numbers in the blue cells are zero or very small, and smaller than the numbers in the yellow cells. The one exception is the parity feature, where 4 students who had referred to this in phase 1 no longer did so in phase 4; however, it can be argued that this feature is relatively unimportant - along with the 'increment' feature perhaps (which is contingent on particular sets of numerical examples); moreover the phase 4 responses of each of these 4 students contained at least one of the other features for the first time ('times 3', 'compensation' or 'algebra').

Thus, we would argue that there is clear evidence of progress in the quality of students' responses in the course of this 4-phase homework and lesson sequence, even if the progress is relatively modest in some key aspects of proof (eg in the development of structural arguments).

Of course, we do not know how long these effects lasted, or whether students would have made further progress if they had continued to work in this way. And learning to work effectively in this way is far from unproblematic, involving amongst other things complex group dynamics, and the need to develop exposition skills on the part of the students and intervention skills on the part of the teacher. However, bearing in mind that the students were not used to working in this way, and that the teacher deliberately held back in clarifying or promoting particular observations and arguments, the changes in the student responses are encouraging and suggest that such an approach has considerable potential.

NOTES

1. Interestingly, the current UK Secretary of State for Education has just announced (October 2006) the imminent withdrawal of coursework from GCSE mathematics - though not for mathematical reasons.

2. All the students in the class except one gained a GCSE pass in mathematics at grade C or above in the following year. Nationally, most students take this examination, and of these just over 50% achieved such grades.

Acknowledgements

We are grateful to the DfES for funding this work and to the groups of teachers and advisors in Hampshire and Suffolk with whom we worked.

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