REPRESENTING MULTIPLICATION

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In this paper, we examine the importance of representations, in particular with respect to the understanding of multiplication by primary school pupils. The first half of the paper examines the theoretical background to representations in mathematics. The second half of the paper examines some preliminary work that we have carried out, examining Year 4 and Year 6 pupils' use of the array representation for multiplication calculations. Using a novel methodological approach of recording children's workings on a computer, we observed that the array representation can be a powerful tool for supporting work in multiplication. At the same time, we also observed pupils who were unable to access the mathematical meanings of the representation. We must be aware of such difficulties when developing the use of the array as a tool for mathematical understanding.

INTRODUCTION

During primary education, pupils are introduced to a number of big ideas – for example addition and subtraction in Key Stage 1, and multiplication and division in Key Stage 2. In teaching addition and subtraction, there has been a clear use of representations such as number squares and number lines, with number lines being the most appropriate representation for demonstrating the characteristics of these operations. In the forthcoming renewed Primary Framework for literacy and mathematics (DfES, 2006), teachers are encouraged to use the array representation for multiplication when introducing pupils to the distributive law. However, how will children respond to this particular representation? This is the question that we will examine in this paper.

ROLE OF REPRESENTATIONS IN MATHEMATICAL LEARNING

Researchers such as Gray et al. (1997), Tall (1993), Krutetskii (1976), Carpenter et al. (1999), Gravemeijer (1994) and Thompson (1999) have all discussed the importance of representations in developing mathematical competence. Representations in mathematics education refer to both internal and external manifestations of mathematical concepts. Hiebert and Carpenter (1992) state that communicating mathematical ideas requires external representations (so spoken language, written symbols, pictures or physical objects), whereas to think about mathematical ideas requires internal representations. Goldin and Shteingold (2001) also identified computer-based microworlds as external representations, and students' personal symbols, their natural language, visual imagery, spatial representation and problem-solving strategies and heuristics as internal representations.

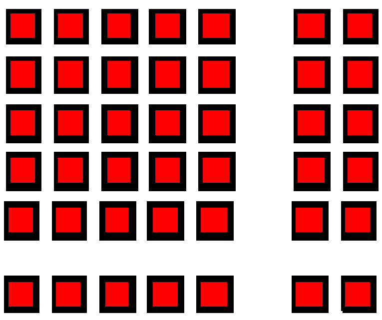
There are a variety of ways in which external representations can be used in the learning of mathematics. First of all, concrete representations can be used to 'mirror' the structure of a particular concept (Boulton-Lewis, 1998). In turn, they are easier to talk about than language-based or symbol-based procedures, and also allows the teacher to access, to some degree, the cognitive processes of students (Hall, 1998). Representations can be used in problem solving situations to make sense of situations and to structure thinking and approaches to solutions (Fennell and Rowan, 2001).

In addition to being helpful tools, researchers have suggested that representations play a more profound role in the learning of mathematics. Lesh *et al.* (1983) suggested that "to do mathematics is to create and manipulate structures" (p.268). During problem solving, subjects "frequently change the problem representation from one form to another" (p.264). In addition to external representations, this creation, manipulation and changing between will involve internal representations. In fact, Hiebert and Carpenter (1992, p.67) state that "mathematics is understood if its mental representation is part of a network of representations." In other words, the creation and switching between representations is central to the understanding of mathematics.

USING ARRAYS TO REPRESENT MULTIPLICATION

Nunes and Bryant (1996) state that a commonly held view of multiplication and division is that they are simply "different arithmetic operations … taught after they have learned addition and subtraction" (p.144). However they suggest that in actual fact "multiplication and division represent a significant qualitative change in children's thinking" (p.144). Whilst addition and subtraction can be thought of as the joining of sets, multiplication is about replication. Anghileri (2000) concurs with these ideas and suggests that one of the problems for pupils when they work on multiplication is that they view it as a unary operation with one input, the second number being a means of identifying a termination point. However, she suggests that pupils need to view multiplication as a binary operation with two inputs. The first input represents the size of a set and the second represents the number of replications of that set. Thus the two numbers represent distinct elements of the multiplication process.

In order to encourage pupils to develop their thinking about multiplication as a binary operation, we contend that the array representation needs to be used, with rows and columns representing the two inputs (see opposite). Also, the array representation is advantageous in that it a useful for showing a number of properties concerning multiplication. It is clear that the array representation shows the commutative property of multiplication (Anghileri, 2000). Swapping the '7' and the '6' in the figure simply changes the orientation of the array. Also, the representation (especially with the additional spaces inserted) makes clear how to differentiate the two numbers being multiplied (each dimension) and the total. We cannot swap that '42' for the '6' or the '7'. The representation also makes clear why the distributive law applies to multiplication. 7 x 6 is equal to 7 × (5 + 1)



$7 \times 6 = 42$ or $6 \times 7 = 42$

or (5 + 2) × (5 + 1). Thus, the 7 times table could be seen as the (5 + 2) times table. Each dimension of the array can be split up into its constituent parts and the representation leads naturally to the grid method.

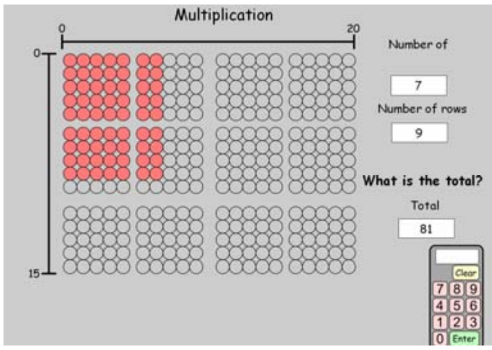
PUPILS ADOPTING REPRESENTATIONS

Having established the usefulness of the array representation for multiplication, how can we encourage pupils to adopt and use this representation? Looking at representations generally, Cobb, Yackel and Wood (1992, p.4) describe one approach, which they termed the instructional representation approach, as "to help students construct mental representations that correctly or accurately mirror mathematical relationships located outside the mind in instructional representations". External instructional representations which are 'transparent' in their use and the mathematics that they embody therefore form the basis for such an approach. However, this type of approach has been criticised for assuming that learners can access and understand the mathematical concepts inherent in the external representations. Problems may occur "from the students' lack of opportunity to explore the structure of new kinds of problems before being asked to apply new symbol manipulation skills. Thus students are being asked to solve problems before they really understand them." (Brenner et al., 1997, p.666) We should rather consider the act of constructing internal representations as a more dynamic process (Cifarelli, 1998), that takes places as learners try to make sense of mathematical situations in the broader, social, classroom context (Pape and Tchoshanov, 2001). Therefore, in order to examine how learners construct and utilise representations, "the challenge is to explain how students construct their mathematical ways of knowing as they interact with others in the course of their mathematical acculturation" (Cobb *et al.*, 1992, p.17).

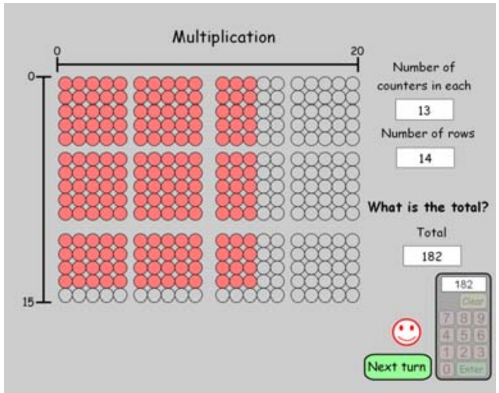
EXAMINING PUPILS USE OF THE ARRAY REPRESENTATION

Having examined some of the theoretical issues involved, let us now look at two case studies where children's use of the array representation was examined. These case studies come from research that we have carried out involving a Year 4 class and a Year 6 class in a primary school in the North East of England. The classroom sessions were based around pairs of pupils using a Macromedia Flash computer program, incorporating the array representation of multiplication. The reason we used a computer program for children's work is that it enabled us to use a novel methodology for examining children's mathematics. During the working in pairs and using the Flash program to do problems on the computer, all the actions carried out by the children on the computer were recorded using Camtasia recording software. As children were each wearing a headphones and microphone set, the dialogue of children as they worked on the multiplication problems was also recorded along with their computer actions. Therefore, a rich amount of qualitative audiovisual data, showing the dynamic process of children using the array representation, was obtained. This therefore enabled us to observe pupils as they "construct their mathematical ways of knowing as they interact with others" whilst they used the array representation for multiplication. We provide here two instances from the data that provided some insight into how pupils were using the array representation.

James and Tom (Year 6)



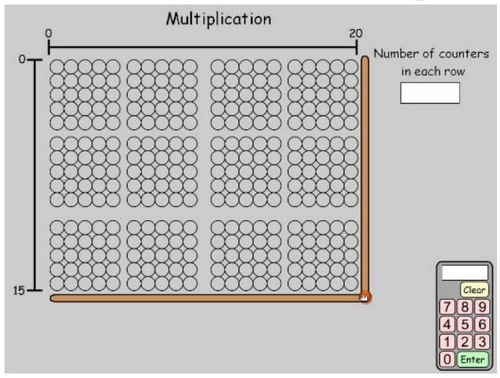
James and Tom were using a programme which asked them to do a multiplication calculation using the array. The numbers were inputted as rows and columns, and then we were interested in how they would do the calculation. The two diagrams illustrate the strategies used. First the pupils focussed on the 25 block and used that as a starting point for counting, using the language “that’s five 5s” and sometimes using the pointer to indicate the block. Then they filled in blank counters to make up another block of 25. For example, in the figure above, by moving one line of 5 down to the bottom left hand block. Finally they counted the rest as a 5 and four 2s. The second example shows what could be a more sophisticated strategy which uses the distributive law. This time, the conversation went as follows:



> Got 100, if I add one over there; 50, 50, 100. And then 4 times 5 equals 20. And another. 3 times 5 equals 15; and 3 times 4 is 12; so two 15s, 12, and two 20s, and 100. So that’s 182.

This time, not necessarily intentionally, the pupils have calculated (5+5+4) × (5+5+3) by taking appropriate sections of the array.

Anna and Beth (Year 4 pupils)



Anna and Beth were using a program where we had asked them to use an ‘L-shape’ with an array (see figure opposite) to do multiplication problems. First of all, they were asked to do the problem 11 × 6. They knew the answer to be 66. However, they then struggled for a few minutes in moving the L-shape so that only 66 squares were enclosed. They did not associate the dimensions of ‘11’ and ‘6’ with the dimensions of the required array. This was despite the fact that they had used a similar program in a previous session, where they typed in the two numbers being multiplied to produce an array of highlighted counters (see the Year 6 example). Eventually, by accident, they placed the L-shape in the correct position and complete the question by writing ‘11’ for the number of rows, ‘6’ for the number of columns, then ‘66’ for the answer. They then tried 14 × 7. They began by trying to calculate the answer first (incorrectly getting

104), then tried to show 104 on the array. They again made no association between the array and the multiplication sum. After about eight minutes of moving the L-shape around the array, one of the researchers intervened.

CONCLUSIONS

On the basis of this small preliminary study, the following points can be made. Firstly, the array can provide a representation of multiplication that not only emphasises the replication element of multiplication, but can also be used to aid the calculation of the total. Secondly, the pupils could be imaginative in their use of the array to calculate the total number of counters. They seemed to see it as a counting exercise and sought ways to make this exercise efficient, by using the 5 by 5 block as a kind of benchmark from which to work. There is a sense, of course, in which this could be seen as a distraction since the calculating exercise then loses the element of replication. Thirdly, the distributive element of multiplication can be used to perform the calculation and this could be an important developmental point. However, we do need to be careful to examine how the pupils see the representation themselves. As in the case of the Year 4 pupils, we cannot take for granted how the children will interpret the representation. Further work now need to be undertaken to gain a better understanding of how pupils make sense of the array in the context of multiplication; whether it encourages the development of informal strategies; whether it provides an important link to the grid method; and whether it can be used to represent a range of word problems.

REFERENCES

Anghileri, J. (2000). Teaching Number Sense. London: Continuum.

Boulton-Lewis, G. M. (1998). 'Children's Strategy Use and Interpretations of Mathematical Representations'. Journal of Mathematical Behaviour, 17(2), pp. 219-237

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Duran, R., Smith Reed, B and Webb, D. (1997). 'Learning by Understanding: The Role of Multiple Representations in learning Algebra'. American Educational Research Journal, 34(4), pp. 663-689

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). Children's mathematics: Cognitively Guided Instruction. Portsmouth, NH: Heinemann.

Cifarelli, V. V. (1998). 'The Development of Mental Representations as a Problem Solving Activity'. Journal of Mathematical Behaviour, 17(2), pp. 239-264

Cobb, P., Yackel, E. and Wood, T. (1992). 'A Constructivist Alternative to the Representational View of Mind in Mathematics Education'. Journal of Research in Mathematics Education, 23(1), pp. 2 – 33

Department for Education and Skills (2006). Primary Framework for literacy and mathematics: core position papers underpinning the renewal of guidance for teaching literacy and mathematics. Norwich: DfES.

Fennell, F. and Rowan, T. (2001). 'Representation: An Important Process for Teaching and Learning Mathematics'. Teaching Children Mathematics, 7(4), pp. 288-292

Goldin, G. A. & Shteingold, N. (2001). 'Systems of representation and the development of mathematical concepts'. In A. A. Cuoco & F. R. Curcio (eds.) The roles of representation in school mathematics. Reston, VA: National Council of Teachers of Mathematics, pp. 1-23

Gravemeijer, K. P. E. (1991). 'An instruction-theoretical reflection on the use of manipulatives'. In L. Streefland (ed.). Realistic Mathematics Education in Primary School. Utrecht: Cdß-Press, pp. 57-76

Gray, E., Pitta, D. & Tall, D. O. (1997). 'The nature of the object as an integral component of numerical processes'. In E. Pehkonen (ed.) Proceedings of the 21st International Conference for the Psychology of Mathematics Education, Lahti: Finland. Vol. 1, pp. 115–130

Hall, N. (1998). 'Concrete Representations and the Procedural Analogy Theory'. Journal of Mathematical Behaviour, 17(1), pp. 33-51

Hiebert, J. and Carpenter, T. P. (1992). 'Learning and Teaching with Understanding'. In Grouws, D. A. (ed.) Handbook of Research on Mathematics Teaching and Learning. New York: NCTM

Krutetskii, V. (1976). The psychology of mathematical abilities in school children. Chicago: University of Chicago Press.

Lesh, R., Landau, M. and Hamilton, E. (1983). 'Conceptual Models and Applied Mathematical Problem-Solving Research'. In Lesh, R. and Landau, M. (eds.) Acquisition of Mathematical Concepts and Processes. London: Academic Press

Nunes T. and Bryant P. (1996). Children doing Mathematics. Oxford: Blackwell Publishers

Pape, S. J. and Tchoshanov, M. A. (2001). 'The Role of Representation(s) in Developing Mathematical Understanding'. Theory into Practice, 40(2), pp. 118-127

Tall, D. (1993). 'The Transition from Arithmetic to Algebra: Number Patterns or Proceptual Programming'. In Proceedings of Second Annual Conference on Teaching and Learning, London.

Thompson, I. (1999). Issues in Teaching Numeracy in Primary Schools. Oxford: Open University Press.