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LEARNERS' SHIFTING PERCEPTIONS OF RANDOMNESS

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In clinical interviews, learners were invited to talk about their experiences of making sense of the emerging sequence of outcomes from repeated trials using different generators, some of which were biased. Analysis of the interviews revealed distinct ways of viewing the phenomena represented by the interview tasks. Drawing upon the local and global meanings of randomness identified by Pratt (1998), learners were found to shift rapidly between local and global perspectives. In this paper, data from a single interview is presented to illustrate the shifting perspectives.

STIMULUS TASKS

When examining people's perceptions and understanding of randomness, a key distinction to make is that between a random process and randomness in a sequence of outcomes generated by a random process (Zabell, 1992). Previous research into perceptions of randomness has not always been clear about this distinction (Nickerson, 2002), but some writers have discussed the issue explicitly (Falk and Konold, 1997; Wagenaar, 1991). The position that randomness is a property of a process rather than of the outcomes was adopted by Wagenaar (1991), and related to this is the view that any outcome from a random process is considered a random outcome (Pollatsek and Konold, 1991). However, it is only by observing outcomes from a process that one can judge whether the process is random.

Since my own introduction to statistics and probability at secondary school I have seen randomness as 'dynamic'. For me, a random sequence could not be printed out as a permanent record without losing the essence of what it is to be random. I felt strongly that random number tables were the antithesis of what I understood by "random". The sequence of numbers was the same whenever I opened the book, and I felt a need to invent a 'random' (and dynamic) process for selecting numbers from the page.

I see randomness as a model to describe a process, and as providing explanation for the outcomes observed. The sequence of observed outcomes might be described as 'random' if it were considered to have arisen from a process that is reasonably considered to be modelled by 'randomness'. This is a key aspect of the notion of randomness that I wish to convey to my students.

Commonly used stimulus tasks in previous research may be classified into two categories (Falk and Konold, 1997). Generation tasks require subjects to make up random sequences of outcomes to simulate the outcomes from a random process such as 'tossing a coin'. In recognition (perception or judgement) tasks, subjects decide whether a given outcome was produced by a random process, or select the 'most random' of several sets of results.

Falk and Konold suggest (1997) that recognition tasks "may be more appropriate for revealing subjective concepts of randomness" because "a person could perceive randomness 'accurately' and still be unable to reproduce it" (p302). Indeed there is clear suggestion from many studies (Nickerson, 2002; Shaughnessy, 1992) that people are generally not good at generating random sequences. Typically, people trying to simulate a random process tend to produce fewer long runs, and more alternations between outcomes, than would be expected from a random process (Falk and Konold, 1997). Other studies have used recognition tasks to explore what sequences people consider to be maximally random; these again show that people tend to identify randomness with sequences having an excess of alternations between outcomes (Falk and Konold, 1997).

In recognition tasks, the sequences presented are static, just as a random number table was 'fixed' for me. A person's attention is focussed only on a given sequence of outcomes and they may never consider the process by which these outcomes were generated. Generation tasks encourage a more dynamic view, since the subject may need to consider unpredictability as they 'generate' the sequence, and may be able to recall after the event the assumptions they had about the process. However, tasks requiring real-time interaction with outcomes from a random process would more effectively prompt learners to express their ideas of randomness as process.

METHOD

Eighteen learners, aged from 13 to 17 years, undertook clinical interviews for about an hour each. The study took place in two stages, with nine interviews in each stage. In the first stage, interviewees worked on tasks using three unusual dice: biased, spherical and cracked. In the second stage a new task using sampling bags was introduced. Interviewees were encouraged to talk reflectively about their experiences of working on the tasks and about what they were thinking.

The biased die looks like a standard cube, except it has two faces labelled 5 and no 3. It has a weight in the face labelled 1, so it is heavily biased towards showing 6. The spherical die is a hollow sphere containing a small bead. The sphere is marked symmetrically with numbers 1 to 6 and, when rolled on a flat surface, it always stops with one of the six numbers uppermost. If it is correctly balanced, each of the six outcomes should be equally likely. The cracked cubical die has a split running across the face labelled 6, and spreading partway across the faces labelled 2 and 5. Since interviewees used this die after their experiment with the biased die, I expected them to consider that it might also be biased.

In each experiment, the learner was first asked to comment on the appearance of the die and to consider how it might behave when rolled several times. The learner was invited to roll the die a few times before commenting on the observed outcomes. I encouraged learners to talk about their thinking throughout and I watched their behaviour closely. If a learner appeared to show concern about a run of outcomes, or even an individual outcome, I invited them to explain what they were thinking.

I hoped that using three different dice would increase the learner's awareness of what they expected from each die, and their willingness to articulate their assumptions. In particular, I hoped that the tasks would provoke learners to talk about how to recognise equally likely outcomes and whether these were necessary for the die to be considered to behave 'randomly'.

LOCAL AND GLOBAL PERSPECTIVES ON RANDOMNESS

Pratt's study (1998) of ways in which children aged 10 and 11 years articulated their ideas and beliefs as they worked in a carefully designed computer-based domain, distinguished two categories of meaning for randomness expressed by children. Local meanings were related to uncertain behaviour of the process and were focused on "trial by trial variation", while global meanings evolved as children recognised the importance of observing a larger number of trials and discerned features of distribution in the long run. Pratt saw the transition from local to global meanings in an individual as lengthy and complex. He did not report movement from global to local meanings, and it seems implicit in the tasks he set and the probes he used, that no such switch was either anticipated or looked for.

Analysis of interview transcripts in the present study showed learners thinking about random outcomes using two contrasting perspectives: the local and the global. In the local perspective, attention is on the uncertainty of the next outcome and ephemeral patterns that appear in short sequences of outcomes. The learner does not aggregate outcomes or to think in terms of a distribution. In the global perspective, the learner is aware of a distribution of outcomes, either empirically, as an emerging frequency distribution of observed outcomes, or in terms of prior beliefs about the generating process (for example, when rolling a die, expecting the outcomes to be equally likely). Learners' attention was found to shift frequently, and sometimes rapidly, between these perspectives, from local to global and back to local.

THE DATA

In this paper, I illustrate the rapid shifting between the local and global perspectives using data from a single dice task with the spherical die, in one interview with Ben from the first stage.

In the first seven throws, Ben had observed {5 5 1 4 1 6 1}. His initial strategy was to look for patterns in the sequence.

Ben: …we haven't had any 3s or 2s, so it could be one of those, but – well, it'll probably be another number than a 1.

I: Why?

Ben: Just from following the pattern. If it wasn't a die, that's what I'd say.

Ben noted that, since this was a die, any outcome was possible. He went on to look for an explanation for the absence of 2s and 3s.

Ben: It might be the way I'm throwing it though. Or when I picked it up, I'm throwing it the *same way*. *Or it could just be chance.*

Ben was thinking about the generating process, but still from a local perspective. Out of concern about lack of 2s and 3s, Ben went on to check the labelling of the die. When the fourteenth outcome was 3, Ben cheered!

Ben now attended to physical factors affecting the outcomes. He played deliberately with the die between rolls, and considered the shape of the weight moving inside the sphere. When I asked how Ben would know this was a fair die, he expressed clearly a global perspective based on his prior belief about a fair die.

Ben: …You just have to keep rolling it. It should in the end even out if it's a fair dice. **If it's not a fair dice it'll… keep on staying away from the 2s and 3s**, like it is at the moment.

I: Are you worried about it being fair?

Ben: …No, not really. …**It could just be chance**. If there's a 1 in 6 chance of getting each different number... I just haven't got a 2 yet, which is strange. Although **I'll probably get a 2 now**, if I roll it...

Ben appeared to be rapidly switching between contrasting views about this die. He understood he needed more trials, a characteristic of the global empirical perspective, and he expressed a prior belief about distribution: the die would be fair.

At the local level, he looked for the first occurrence of a 2, and expressed concern that he had not seen it after fifteen throws. At the global level, looking for a frequency distribution to match his prior belief, he accepted "It could just be chance". By changing the focus of his awareness, Ben arrived at two different explanations for the absence of 2.

After 17 throws without a 2, Ben had observed {5 5 1 4 1 6 1 1 5 4 4 5 1 3 6 5 4 5 5}. He was quiet, and he experimented with the die, rolling it in his hand without talking for 13 seconds, before commenting

Ben: It seems pretty fair. But it depends what happens when you roll it

He seemed to experience a tension between apparent 'fairness' of the process, and imbalance in the outcomes.

On the next throw, Ben rolled a 6, but he wanted the die to show a two. It was as though he wanted to remove the anticipation of waiting for a two to occur, and by experimenting with the way he rolled the die he was trying to make it happen.

Ben: …oh land on a 2.

On the next throw Ben rolled the die, and got a 2! He was excited and began to think he could control the outcomes. His experimenting with how to roll the die was associated in time with rolling a two. This supported his idea that the way he rolled the die controlled individual outcomes.

Ben now went quiet again, until I asked him what he was thinking about. He commented that the axis of rolling the die did not explain the outcomes as 2 and 3 were not opposite to each other on the die.

Ben: Just seeing… if I was always rolling it in a way so it only lands on 6, 5 1, 4. But that wouldn't work, or make sense… it stays away from 2s and 3s, but it won't cos they're not next to each other – but they are.

Here again he was reasoning a local perspective, trying to find an explanation for the short sequence of outcomes observed. But in the next sentence he switched to expressing a global explanation.

Ben: It might be weighted more heavily on the 2 and the 3, on the inside, I was just thinking. If the weight's heavier there it will be less likely to turn that way.

Ben rolled a 1 and remarked that the die seemed more random now.

Ben: The more you do it, you know, the more different… But at the start it was all the same. So the more you do it the better the results you get, I suppose.

He found a possible global explanation and he tried to stabilise this idea in his mind. The next throw produced a 5.

Ben: …It should, unless it's weighted, be completely random. But at the start it just seemed to be 5s and 1s. But then it… just got a lot more mixed as it went down, so I suppose… it's just… more and more of a dice and, sort of less chance that the odd number will count for so much. You got a couple of 5s at the beginning, then, later as you go on, you'll get more of the other numbers as well. In theory, I think. …Although I haven't got that many 2s still.

As he moved towards a stable global perspective, Ben was holding in tension the two contrasting ideas of randomness (by which he means equiprobability), and bias – and he expressed them alternately. These were the apparently conflicting global interpretations for Ben: prior belief and a global frequentist view, possibly emerging from the aggregation of the observed outcomes.

As soon as he had expressed the idea of randomness, Ben reverted to discussing the bias. He rolled another 5 and reverted to a local perspective.

Ben: But I have got quite few 5s I think. …But that could just be the way I'm rolling it.

Over the next few throws, Ben's concern about bias diminished as he obtained more 2s and 3s. He remarked again on the apparent randomness.

Ben: Maybe it's just… I suppose it could just be a completely fair dice… It does have quite a few 5s, but that just might be me rolling it, rather than the dice would be weighted or something.

CONCLUSIONS

Ben's ideas were strongly affected by short run behaviour of the die. When the sequence of recent outcomes did not include one or two of the possible outcomes, he tried to explain the apparent bias. When the missing outcomes had appeared once or twice, he described the behaviour of the die as "random". Sometimes "random" was "the absence of pattern", and this cue was switched on and off by short-term changes in the sequence of outcomes.

Ben's interpretation of the outcomes was also influenced by the fact that he did not know how much variability to expect from a fair die. For example, he did not know how many throws he might need to wait until all six outcomes had appeared at least once, or how often should the most commonly occurring outcome appear in the first *n* throws of the die. Therefore Ben could not judge whether he had seen too many 5s, or whether the waiting times he observed before the first 3 and the first 2, were appropriate in a fair die. To refine his judgement of whether a die was fair, Ben needed intuitions about variability. Understanding of variability is important in reconciling the local and the global views of randomness.

In all the interviews, shifting attention between local and global perspectives was common, and the phenomenon seemed to be fuelled by a desire to draw conclusions from short sequences of outcomes. This in turn seems to be related to a poor understanding of variability.

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