PEDAGOGICAL SENSITIVITY AND PROCEDURAL THINKING: AN UNEASY RELATIONSHIP?

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We examined the relationship between subject-matter knowledge and pedagogical content knowledge of 53 in-service mathematics teachers in the context of their written responses to a question that involved: solving the equation $|x|+|x-1|=0$*, examining a flawed student solution and providing feedback to the student. Here we focus on a group of scripts characterised by pedagogical sensitivity but constrained mathematically (substantively and meta-cognitively). Through examination of examples from the data we demonstrate, and discuss implications of, some of these constraints: insistence on standard procedural methods, inappropriate contextualisation of otherwise commendable pedagogical practices and inadequate reflection on student thinking.*

Work on how the preparation of teachers can facilitate the development and employment of their subject-matter knowledge and beliefs and knowledge about pedagogy gained significant theoretical momentum from the 1980s onwards – particularly with Shulman's (1986, 1987) now seminal seven-type taxonomy of teacher knowledge. Crucially this taxonomy included *pedagogical content knowledge*, 'the ways of representing the subject that make it comprehensible to others' (1986, ibid, p9). The 'most useful forms of representation' of mathematical ideas as well as an understanding of students' background, previous knowledge, preconceptions and potential difficulties were included in Shulman's definition. Latter-day explorations of the manifold nature of teacher knowledge highlighted *sensitivity to students* and *mathematical challenge* (Jaworski 1994) as aspects of teachers' practice where their pedagogical content knowledge is most evident. Furthermore this and other studies – for example, see Thompson (1992) for a review – acknowledge that, given the overt discrepancy between theoretically and out-of context expressed teacher beliefs about mathematics and pedagogy (e.g. in interview-based studies) and actual practice, *teacher knowledge is better explored in situation-specific contexts*.

Our work explores teacher knowledge in such a context. In particular the study we report here[1] explores how pedagogical practice is influenced by ways of knowing about mathematics (e.g. in the light of Mason and Spence's (1999) *knowing-to act mathematically*. The particular examples from the data we employ in this paper are also seen in the light of the distinction between *procedural and conceptual mathematical thinking* (Hiebert, 1986).

[1] An ongoing small-scale study conducted by a team of researchers based in Greece and the UK and supported by an ERASMUS grant and by the University of Athens (ELKE). The data presented here have been translated from Greek. The team consists of three mathematicians with complementary backgrounds: a university lecturer and researcher in mathematics education; a mathematics teacher currently completing a doctorate in mathematics education; and, a university lecturer and researcher in mathematics with a currently developing interest in mathematics education research.

THE STUDY

Context and Data Collection. The data we draw on in this paper are the written responses to one (Fig. 1) out of nine questions set in an exam taken by 53 in-service secondary mathematics teachers for the purpose of selection for a Masters in Mathematics Education programme. All participants are mathematics graduates with teaching experience that ranges from a few to many years and most have attended in-service training of about 80 hours.

In a mathematics test students were given the problem:

"Solve the equation: $|x|+|x-1|=0$"

a. What do you think the examiner intended by setting this problem?
b. A student responded as follows:

"It is true that

$$|x|+|x-1|=0 \Leftrightarrow \left(|x|+|x-1|\right)^2=0 \Leftrightarrow x^2+\left(x-1\right)^2+2|x(x-1)|=0 \Leftrightarrow$$
$$x^2+x^2-2x+1+2|x(x-1)|=0$$

Case 1: $x(x-1)\leq 0$

Then

$2x^2-2x+1-2x^2+2x=0 \Leftrightarrow 1=0$ Impossible.

Case 2: $x(x-1)\geq 0$

Then

$$2x^2-2x+1+2x^2-2x=0 \Leftrightarrow 4x^2-4x+1=0 \Leftrightarrow \left(2x-1\right)^2=0 \Leftrightarrow x=\frac{1}{2}$$

Therefore the solution of the equation is $x=\frac{1}{2}$."

What comments would you make to this student with regard to this response?

Fig. 1

In the above the student has successfully carried out a case by case study of the equation but has failed to observe that in the second case ½ is not a solution (it does not satisfy $x(x-1) \geq 0$) and therefore the equation has no solutions. Furthermore the student has not observed that, instead of this acceptable but long-winded procedural approach (which, along with the approach that distinguishes cases according to the sign of x and x-1, we will call the *'routine'* methods), the above conclusion can be reached by simply noticing that, given the non-negative nature of absolute value (| |), $|x|$ and $|x-1|$ must be simultaneously zero if they add to zero – which is impossible.

We call this latter approach the *'optimum'* solution as it is shorter and draws on a deeper conceptual understanding of | |: selecting a crucial attribute of the entity, its non-negativity, and, through mathematical logic, applying it towards the solution of the equation. Also, from a didactical point of view, acknowledging this solution offers an opportunity to discuss meta-cognitive issues such as benefiting from awareness of multiple ways of approaching a problem. Of course the procedural approach has advantages as well: it is generalisable and demonstrates substantial ability in algebraic manipulation. Furthermore teachers' preference for this approach may be grounded on the idea that at this stage of their students' learning it is too confusing to bombard them with alternative solutions when they are still struggling with the standard methods[2].

The objectives for including this question in the exam[3] were:

- To assess the candidates' subject-matter knowledge.
- To assess the candidates' sensitivity to student difficulty and needs; and,
- To assess the candidates' ability to provide adequate (pedagogically sensitive and mathematically precise) feedback to the student.

With regard the second and third of the above in part (b) of question the candidates were expected to explain how they would work with the student towards: casting doubt on whether ½ is a solution to the equation; identifying where the student's case by case exploration was flawed; reconstructing the student solution; observing that the student may have chosen an unnecessarily long-winded approach whereas other approaches may exist; suggesting the 'optimum' solution; and, juxtaposing and discussing the merits / weaknesses of each of the two solutions and the value in circumventing routine methods in favour of concise and elegant ones.

The candidates' responses were expected to operate at least at three levels: the substantive (algebraic and logical manipulations in solving equations; conceptual understanding of | |), the meta-cognitive (acknowledgement of the multiple ways in which an equation can be solved; optimal choice of solution) and the didactical (utilise the opportunity offered by the problem to discuss problem solving skills such as the above mentioned elements of meta-cognitive awareness). We perceived a candidate's preference and emphasis on the 'optimum' solution to be evidence of mathematical subtlety combined with sensitivity towards the learning situation in question.

The strength and idiosyncrasy of this type of data lies in its highly focused, situation-specific character: candidates are asked to respond to mathematically / pedagogically specific situations that are likely to occur in the mathematics classrooms they are usually operating in. We believe that this specificity (Dawson, 1999) – in contrast to posing questions pitched at a theoretical, decontextualised level – can generate authentic access to candidates' views and intended practices. Of course we also recognise that the teachers were *not* in the classroom and they had some time to think about their reaction. However we consider that the latter may allow teacher responses to be more representative of their *intentions*. We would also like to note a methodological limitation of the data: the candidates' written responses is the one and only source of evidence we have, thus far, regarding their views and intended practices.

[2] We note that in the analysis that follows graph-based solutions have not been considered, mostly because there was no reference to this type of solution in the 53 scripts. Exploring the reasons for this absence are outside the scope of this paper.

[3] There was one set with similar intentions in the context of Calculus but in this paper we will limit ourselves to an exploration of the Algebra question only.

Data Analysis and Findings. Regarding the analysis of the 53 scripts, for each we produced an Analytical Summary, a narrative of approximately 100-200 words, in which we summarised the script's contents and evaluated the candidate's responses in accordance with the above objectives. Further scrutiny of the scripts and the Analytical Summaries led to a three-dimensional taxonomy and all scripts were characterised in terms of their perceived *mathematical, didactical and pedagogical limitations* (five, five and nine categories respectively). For the purposes of this paper we need to focus only on the first two[4]. The numbers in the brackets refer to the number of scripts allocated in each category.

MATHEMATICAL LIMITATIONS

1. Regards flaw of the student response to be the squaring in the first line (13).
2. Regards that in Case 1 $x(x\text{-}1)\leq 0$ needs to be solved – this is unnecessary (7).
3. Does not see the 'optimum' solution (6).
4. Makes technical mathematical mistakes – e.g. in algebraic manipulations (8).
5. Does not identify the flaws in the student's response (1).

DIDACTICAL LIMITATIONS

1. Does not reconstruct the student response (3).
2. In reconstructing the student's solution, in particular in order to reject ½ as an acceptable solution, proposes the use of standard procedural methods that are unnecessarily convoluted – e.g. solving inequalities, graphing the intervals in which *x* needs to belong etc. – instead of simply substituting x with ½ in the inequality and seeing it needs to be rejected. (21).
3. Presents a routine method other than the student's – see below Fig.1 (2).
4. Despite identifying the 'optimum' solution in part (a), does not refer to it in part (b) – either at all (14) or faintly (3).
5. Describes an overly general and theoretical pedagogical approach (7): offers substantively and meta-cognitively rich propositions but does not embed them in the specific situation set in the question (5 of the seven); alludes to constructivist ideas such as encouraging the student's own reconstruction of the solution but in fact simply letting the student unguided and possibly lost (1); offers limited feedback based on superficial generalisations on student's ability (1).
6. Appears to aim at the use of commendable pedagogical practices, such as exemplifying, but employs them unsuccessfully – e.g. proposes examples that are incorrect, miss the point or are potentially misleading (3).
7. Uses mathematical (terms, symbolism) or ordinary language problematically (8).
8. Focuses excessively on insubstantial, trivial aspects of the question (2).
9. Uses mathematical formalism in an over-the-top and potentially misleading way (1).

[4] For information, under Pedagogical Limitations (P categories) we listed: presenting solution without encouraging participation / discussion (21); no attempt to 'psychoanalyse' the student response (10); lacking meta-cognitive reflection (21); mere identification and correction of the mathematical flaw of the response (13); and, drawing hasty, largely unfounded and clichéd generalisations about the student's ability (3). We would like to stress that we hesitate to draw more definitive inferences from the contents of the P categories for the following reason: most P categories highlight the *absence* of a certain reference in the candidates' response –for example, absence of a reference to encouraging student participation. We feel that we cannot infer a candidate's conscious choice against student participation from this non-reference in the script. To draw such an inference we would need further and more solid evidence, e.g. from observing their classroom practice or interviewing. On the other hand we feel more confident towards acknowledging commendable pedagogical intent to scripts where there are overt references to, for example, encouraging student participation.

Here we wish to focus on the choices made by the candidates in about a fifth of the scripts that we characterised as demonstrating pedagogical sensitivity but were constrained mathematically (at the substantive or meta-cognitive level). In particular we wish to examine how these constraints may divert the candidates away from materialising their good pedagogical intentions. Our concern is not only with their substantive constraints (such as M1, M2, M4 and M5). We are mainly concerned with how insistence on the routine methods (D2 and D3) may have diverted the candidates from thinking about (M3) and/or suggesting to the student (D4) the 'optimum' solution – see exam-setters' expectations earlier.

We identified three types of the above mentioned constraints in this section of the scripts: *insistence on standard procedural methods, inappropriate contextualisation of otherwise commendable pedagogical practices* and *inadequate reflection on student thinking.* Five of the scripts exemplify these constraints more overtly: the scripts of candidates [23], [43], [25], [45] and [50]. Due to limitations of space we present one of these, the script of candidate [25], and summarise the rest. Candidate [25] expressed a preference for commendable general pedagogical practices but in inappropriate ways. She wrote in part (b):

> Depending on the case of the student, I might have started from making him identify his mistake by making him work in the same way but on a simpler exercise such as $|x-1| = 0 \Leftrightarrow (|x-1|)^2 = 0$ ... etc.

Candidate [25] expresses a preference for engaging the student in a process of self-realisation but she attempts to do so with an irrelevant example which doesn't help the student identify his[5] mistake: it can be solved without distinguishing cases and it has a solution. She adds: "To another student (in the case of a student who has the knowledge but likes to complicate otherwise easy things!) I would try to point out the obvious". This comment contains elements of reflection on student thinking and of meta-cognitive feedback to the student. However it is too vague to allow the good intentions to be transformed into useful, concrete suggestions. The candidate felt the need to use a problem or a simpler exercise as a pedagogical tool to help the student understand his error. But the choice of example is inappropriate, even though technically correct. It seems that she could not translate good pedagogical intent into a mathematically coherent and didactically effective suggestion.

We identified analogous tendencies in the scripts of candidates [45] and [50] who place great emphasis on the routine method and choose an example that is supposed to support the refutation of the student's error. Unfortunately their choices are based on an inaccurate judgement regarding the origin of the error (M1). In the occasion these mathematical inaccuracies stall the effect of their otherwise good pedagogical intentions (preference for a dialogue with the student; employment of examples). Finally candidates [23] and [43] devote almost the entirety of their response to procedural aspects of solving the inequality thus diverting students from the conceptual and meta-cognitive aspects of the exercise.

[5] We refer to the candidates as 'she' and to the student in question as 'he'.

PEDAGOGICAL SENSITIVITY TRAPPED AND DIVERTED?

The above scripts were produced by mathematics graduates with some teaching experience and with some interest in reflecting on pedagogical issues. In the cases we examined here briefly there is evidence of positive but stalled pedagogical intent. The main characteristics of the candidates' problematic subject-matter and pedagogical content knowledge were: insistence on standard procedural methods; inappropriate contextualisation of generally commendable pedagogical practices; and, inadequate or inaccurate reflection on student thinking. We highlighted especially how these may impede the didactical effectiveness of the candidates.

Twenty one responses were allocated in D2. Why do these candidates turn to these correct yet unnecessarily, in our case, convoluted approaches? Because their mathematical thinking lacks the flexibility and creativity suggested by less routine (perhaps instigated by mathematical training that is inadequate in this respect)? Because their thinking – mathematical and pedagogical – gravitates towards a deeply ingrained belief in standard, routine methods (perhaps instigated by extensive involvement with, generally, procedurally-based school mathematics)? In any case, despite 'knowing-about'[6] the subject, they are in need of a transition towards 'knowing-to' act.

> Knowing to act when the moment comes requires more than having accumulated knowledge-about. It requires relevant knowledge to come to the fore so that it can be acted upon; [it] requires awareness …[It] is working on this awareness which provides the fulcrum for professional development' (Mason and Spence, ibid, p139).

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[6] Knowing about: an overarching type of knowledge that covers Shulman's categories and includes knowing-that (factual knowledge), knowing-how (technique and skills) and knowing-why (explanatory and reconstructive fluency).