PROBING UNDERSTANDING THROUGH EXAMPLE CONSTRUCTION: THE CASE OF INTEGRATION

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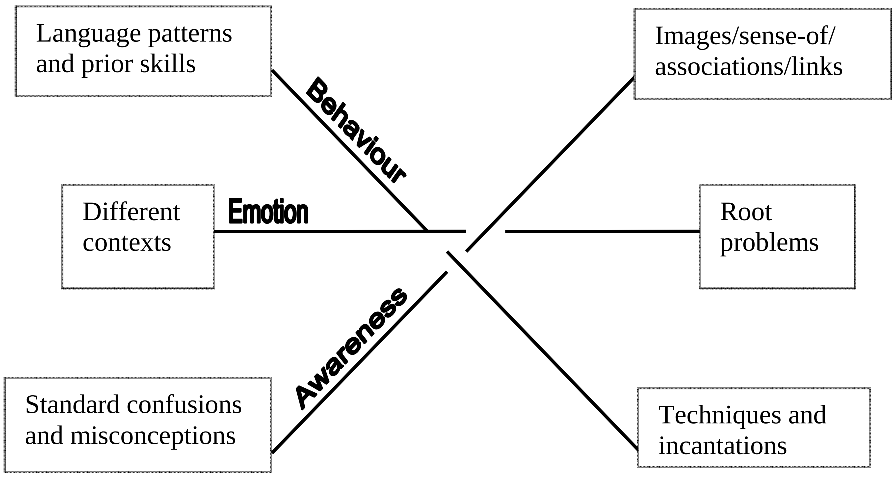
Activities that reveal something about learners as they become aware of aspects of mathematical concepts that were not previously focussed upon can be very useful for the learners themselves as well as for teachers and researchers. In this paper I consider example construction tasks as research probes to reveal learners' awareness of the concept of integration. Forty students studying A-level, engineering, mathematics and education have been invited to construct relevant mathematical objects meeting specified constraints. What learners choose to change in mathematical examples reveals the dynamics and depth of their awareness and acts to promote and enrich their appreciation of the concept.

INTRODUCTION

Coming to understand mathematical ideas can take different forms for different learners. However, a common feature of understanding seems to be that it is highly situated, in the sense that learners may be well-equipped to work on standard textbook problems that appear in familiar contexts, and yet become incapacitated when faced with novel situations. Learners do not seem to be able to cope with situations beyond the familiar. Knowing to act in the moment (Mason & Spence, 1999) requires awareness that brings knowledge relevant to the situation to be acted upon. Thus, I see conceptual understanding as evidenced by behaviour which not only uses knowledge conceived in a familiar setting to solve routine problems correctly, but more importantly, by behaviour which extends that knowledge appropriately and efficiently into unfamiliar situations. Dealing effectively with novel situations depends on which aspect(s) of the concept/idea become(s) the focus of learners' attention, thus regarded as important. In this paper, I explore learners' awareness and understanding of integration by having them construct relevant mathematical examples meeting specified constraints.

Knowledge and Understanding

Understanding of a mathematical topic involves learners getting a sense of it in relation to their past experience. Tall & Vinner (1981) use the term *concept image* to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." Based on Gattegno's (1987) assertion that awareness is what is educable and enriching the notion of personal *concept image* with the three interwoven dimensions of human psyche, a framework referred to as 'structure of a topic' (Mason & Johnston-Wilder, 2004) was developed to describe how a mathematical topic is conceived. It is used to analyze learners' understanding of mathematical topics, as well as to prepare one to teach a topic.



'Structure of a topic' framework (Mason & Johnston-Wilder, 2004)

The framework encompasses three strands: behaviour, emotion and awareness which are closely associated with the more familiar terms *enaction*, *affect* and *cognition*. Behaviour is trained through practice but training alone renders the individual inflexible. Flexibility arises from awareness which informs and directs behaviour. Learning then involves educating awareness which in turn directs appropriate behaviour. Energy and motivation to learn arise from the harnessing of individuals' emotions. Marton (Marton & Booth, 1997) regards learning as making distinctions, both discerning something from, and relating it to, a context. A particular form of active awareness is discernment of variation. This fits with a view of mathematics as being essentially about the study of *invariance in the midst of change*. In the context of learning mathematics, Watson & Mason (2005) suggest that learning involves extending awareness of *dimensions of possible variation* associated with tasks, techniques, concepts and contexts, and extending awareness of the *range of permissible change* within each of those dimensions.

Example construction and awareness of dimensions-of-possible-variation

An integral part of effective mathematics instruction is the use of examples to illustrate and clarify mathematical concepts. While teachers may be attending to the generic aspects of examples, learners may develop restricted thinking that only those particular examples are appropriate, unless explicit attempts were not taken to draw learners' attention specifically to what is learned. Getting learners to construct examples can reveal aspects which dominate learner attention and thus, what they regard as important. By prompting learners to construct mathematical examples, what they choose to change reveals dimensions, depth and scope of their awareness. Although learners do not often encounter this type of question in their normal learning environment, I conjecture that the task itself could offer opportunities to experience structure of mathematical examples, to discern what is invariant and what can be varied and more importantly, and so to reveal their awareness of mathematical concept.

METHOD

Forty students were interviewed in pairs with regard to their understanding of integration, eighteen of which were first-year undergraduates studying mathematics, engineering or PGCE mathematics in three universities in South Midlands and the remaining 22 were studying A-level in a school in the same area. The aim was to reveal the range of responses regarding the understanding of integration in different learners. The semi-structured interviews were tape-recorded and transcribed verbatim. The students were also invited to construct relevant mathematical objects meeting specified constraints with the aim of revealing their awareness of the topic. Learners' responses were analyzed to find out aspects of the topic that were the focus of their attention and how shifting their attention to what must remain invariant and what can vary reveals the structure of their awareness.

DATA ANALYSIS

Analysis of responses from Jim and Dinah (pseudonyms), first year PGCE Mathematics students, to questions on understanding of integration suggests that their responses were highly dominated by behavioural aspect of integration, namely techniques. They displayed fluency in language patterns and techniques of integration.

Researcher: What does the word "integration" mean to you?

Jim: Reverse of differentiation, summation, working out areas, volumes, moments of inertia.

Researcher: Do you have any images/conversations when you see the sign?

Jim: Apart from use in area under a curve and the first principle, but then you quickly get on to the rules and different techniques.

Dinah: That is an integration question, you have to apply particular rules to find [the] value.

Both Jim and Dinah displayed awareness of things to watch out for when doing integration questions, in which they stressed technical aspects of integration.

Researcher: What sorts of things have you discovered you need to watch out for when you are doing integration?

Jim: You've got to work out what rule/algorithm to apply to a particular question. For example integration by substitution, the formula for substitution, the kind of complication you don't get in differentiation.

Dinah: Need to look for limits because your answer differs when you got limits.

The following task was then given to find out about the nature of their awareness and the focus of their attention when working on a problem:

Given that $\int_0^2 (1-x)dx = 0$. Can you find another example like this where the answer is 0?

Jim constructed examples by changing the upper limit and the function while Dinah changed just the function.

Jim: $\int_0^3 3 - x^2 dx$. Dinah: $\int_0^2 (x-1)dx$.

Jim's attention seems to be focussed on the upper limit, increasing it by 1 and changing both terms in the function. He seems to be focusing on the relationship between the limits and the terms in the function. Dinah constructed her example by simply reversing the order of the terms in the function and maintaining the limits, which suggests a limited awareness of or confidence about what can vary. When asked to give another example, both worked on Jim's example, changing the upper limit and the function.

Researcher: Can you give another example?

Jim: I guess $\int_0^4 (4 - x^3)dx$. I haven't worked out the general rule.

Dinah: No, it doesn't work. [Checks]

Jim: It would have to be … 16. Make that *any* number you like. So 16*x* … [calculates]

Dinah: No, it should be $x - 4x$. Or you can make it $\int_0^4 (x - 4x^3)dx$. It won't work.

Jim: $16 - x^3$, that's alright. [Checks] $16 - \frac{x^4}{4}$. Yeah … it will work if you do multiples of 4. So then we'll have $25 - x^2$ … If you do $\int_0^5 (25 - x^4)dx$ it works. [Checks] 25 - $\frac{5^4}{4}$. No, it has to be another multiple of 5 … 125.

Researcher: Can you give another example?

Jim: $\int_0^6 (6^4 - x^5)dx$. Dinah: [Checks] Yeah.

When probed to give another example, Jim increased the upper limit and changed the terms in the function. He discerned the limits as a variable dimension. He seemed to be *guessing* at a pattern between the upper limit, the first term and the second term, admitting that he has not worked out the general rule for the pattern. Having been corrected by his partner, Jim then *adjusted* the first term to accommodate the changed upper limit. Based on his first two examples, Jim *looked for pattern* and generalized (partially correctly) for the third and subsequently, *spotted the pattern* and *generalized*. Dinah checked her generalization by constructing examples consisting of functions $y = x$ and $y = - x$ from -1 to 1 and suggested that a general example could have one positive and one negative limit.

Researcher: Can you give me a general example for which the answer is zero?

Dinah: $\int_{-1}^{1} x dx$ [Checks]. $\int_{-1}^{1} -x dx$, let me find out …is that right?

Researcher: What is the most the general example you can think of for which the answer is zero?

Dinah: If you make one negative and one positive it might work.

Researcher: What can you change?

Jim: The limits and the expression.

Researcher: What happens when you change the limits or the expression?

Dinah: When you change [upper] limits, take multiples of limits. For example 5 [you take] 125.

Jim: So that would be $\int_{0}^{n} (n^{n-2} - x^{n-1}) dx$. You could prove that by induction.

Although both Jim and Dinah displayed awareness of some dimensions that can vary in the example, the fact that other dimensions like variable *x* and method of integration were not mentioned suggested that they did not mark those dimensions as variable and were less likely to appreciate the interconnectedness of these dimensions. This reveals something about the nature of their awareness, which seems to be limited and fragmented.

Although area was mentioned when asked what integration meant, Jim did not display awareness of this association when dealing with integration question. Even though he might be aware of this useful connection, the fact that it was not mentioned when constructing examples like the one given suggests it did not come to mind in the moment. Extra triggers were required to invoke awareness of the integral as area, which called for shifts in attention to seeing connections and relationships.

Researcher: Why is it coming to zero?

Jim: Because this is different area under the curve. [Long pause] We have actually integrated across … We probably got it wrong, have we? $y = 1 - x$ is that line [sketches] So in fact we've got two areas that have been taken away from each other. So in fact the area under the curve… so should we not have integrated from 0 to 1?

Researcher: Now can you think of another example?

Jim: Any value between there [points to limits of *y* = 1 - *x*]. So where you got the area is opposite sides of the *x*-axis. In this case you should integrate it from there to there [points to limits].

DISCUSSION AND CONCLUSION

The different ways in which learners construct mathematical objects can reveal something of the nature and structure of differences in how learners experience and understand what they are learn. Particularly, learners' awareness is revealed through aspects in a problem on which they focus their attention and thus, presumably, regard as important in that moment at least. Getting them to talk about aspects which dominate their attention can reveal dynamics of their thinking. By becoming aware of features not previously at the focus of their attention, learners who appeared to be expecting to be 'tested' about their knowledge of integration in fact revealed to themselves aspects of integration that were not previously salient to them (according to their comments at the end of the interview). Learners can have fragmented awareness of concepts which can differ from what they say and what they do.

The use and exploitation of the 'structure of a topic' framework suggest that there are aspects of a topic that become the focus of learners' awareness and thus central to their attention when doing mathematics. These aspects that are stressed more than others may prevent them from becoming simultaneously aware of other aspects, which would help them gain a richer understanding and appreciation of the topic. The extent to which learners' awareness/attention is drawn explicitly to what is being learned can act to promote and enrich their understanding.

Prompting learners to construct examples not only reveals focal aspects of the object but also their awareness of dimensions of possible variation in mathematical examples so that learners can experience the structure of what is exemplified. Sensitivity of probes is important in revealing more or different things learners could be aware of because different probes get different answers from learners. These probes account for learning about the learners from the probes, as opposed to learning about the probes from learners' responses in other kinds of tasks.

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