SECONDARY SCHOOL PUPILS' APPROACHES TO PROOF-RELATED TASKS IN GEOMETRY

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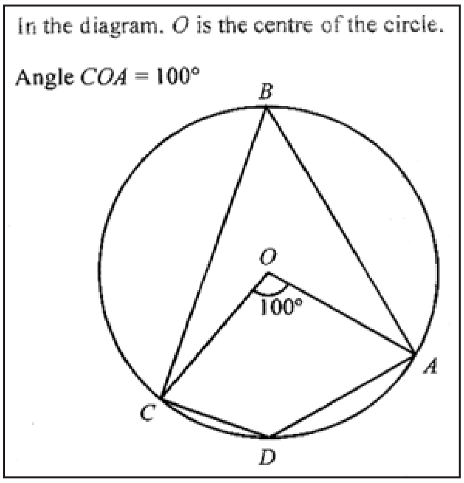
We give an account of two pupils' attempts to solve two GCSE geometry questions involving circle theorems. We identify some of the characteristics of such tasks, some of the pupils' emerging strategies and some of the difficulties the pupils encountered, especially with using the givens and extracting information contained in diagrams.

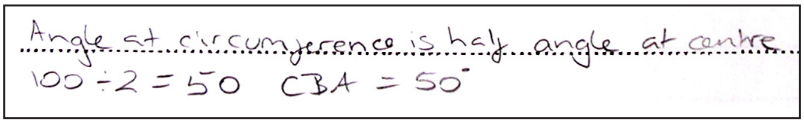
The account in this paper examines two episodes, observed and recorded by the first author, in a geometry lesson with a top set Year 10 class. The lesson began with the teacher giving a quick reminder of some of the circle theorems. The pupils would have met the theorems before but may not have been fluent in using them.

The pupils were then asked to tackle some questions from past GCSE papers. They mostly worked in pairs and we paid particular attention to one pair, Lorna and Kerrie. The episodes bring out a number of issues, including:

1. **The prior knowledge, including theorems, that can be assumed to be true**
   - Different ways of proving the same result, eg, going back to first principles or using other theorems
   - Ordering this knowledge, ie using some relationships and theorems to prove others (systematising)
2. **Determining what information is given in the diagram and text**
   - Interpreting diagrams correctly (not reading information from the diagram that can not be or is not necessarily true, and not making unjustified assumptions)
   - Finding ways of expressing and using what is explicitly or implicitly given.



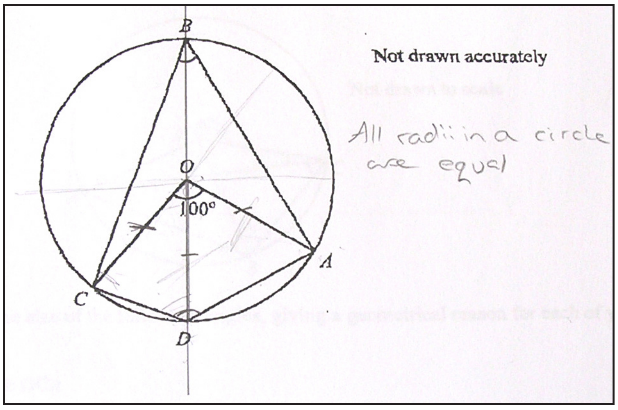
Lorna and Kerrie started with this question (right), in which they were asked to find (a) angle CBA and (b) angle CDA. They had little difficulty with part (a) and Lorna wrote a clear and precise explanation.


Angle at circumference is half angle at centre
100 ÷ 2 = 50 CBA = 50°

However, they did not immediately know what to do for part (b). The item is intended to be solved by using the theorem *Opposite angles of a cyclic quadrilateral are supplementary*. However this was not one of the theorems mentioned by the teacher at the start of the lesson and Lorna and Kerrie clearly did not remember it. After a while, Lorna started almost absentmindedly to add lines to the diagram. First she sketched a horizontal line through O. Then she extended CO. Finally, and more firmly, she drew the radius OD.

It is not clear quite how deliberate and thought-through this construction was. However, it was a significant moment as it made explicit that D is on the circle, so that the property can now be used. Lorna then marked the line segments OC, OD and OA to show that they are the same length and wrote "All radii in a circle are equal". Her diagram is shown below (though the 'diameter' through B, O and D, which obscures her initial construction line OD, had not been drawn at this stage).

Lorna then decided that OD bisects angle COA, which gave her a further way forward. Of course this assumption is not necessarily true. However, when questioned about it, she stuck to her assumption and argued that it could be true even if it didn't particularly look it, since the diagram is "Not drawn accurately". This gave her two isosceles triangles, each with an angle of 50˚ and thus with base angles of 65˚. In turn this meant that the desired angle at D was made up of two angles of 65˚, giving a total of 130˚. This is Lorna's working, which though it lacks an explanation is again clear and quite explicit:

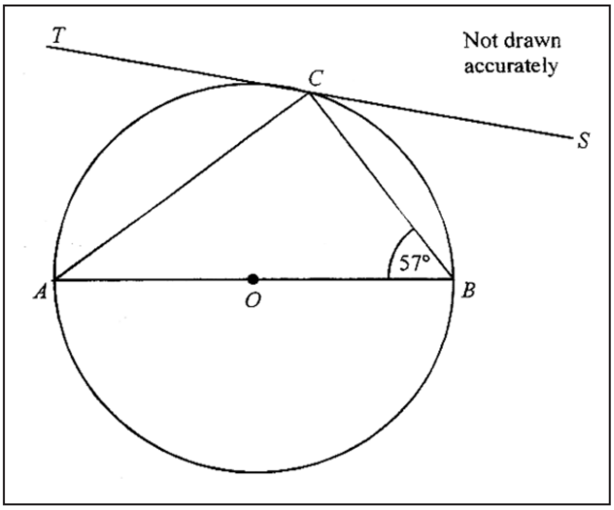


$100 \div 2 = 50 \quad 180 - 50 = 130 \quad 130 \div 2 = 65°$

$65 \times 2 = 130$

To our initial surprise, this method gave the correct value for angle D. It transpires that Lorna's assumption about OD, though she had not justified it adequately and though it is not necessarily true, does not violate the givens: she had hit on a special but valid case. In the subsequent class discussion it emerged that another pupil had similarly found the angle using the unjustified assumption that BD is a diameter - and so in ABCD the angles at A and C are 90˚ and angle D = 360˚ – (50˚ + 90˚ + 90˚).

After the class discussion, Lorna and Kerrie worked on another GCSE question (right). Here AOB is a diameter, TCS is a tangent, angle ABC = 57˚ and pupils are asked to find (a) angle CAB and (b) angle ACS.



Lorna and Kerrie quickly realised that angle ACB is 90˚ (*Angle in a semicircle*) and they solved part (a) without any difficulty:

Angle *CAB*

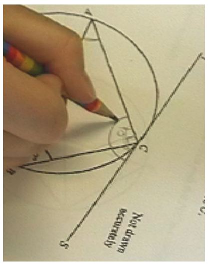
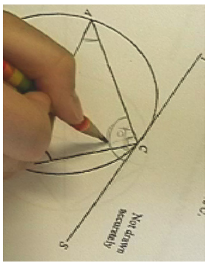
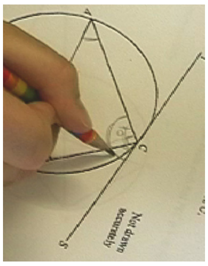
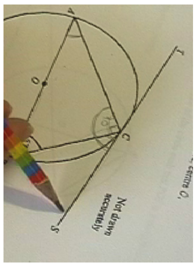
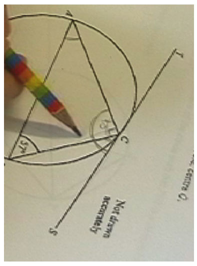
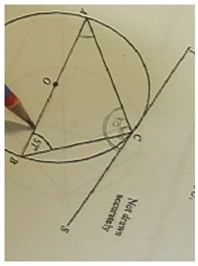
Angle in a semicircle is 90°

$\angle ACB = 90° \quad 90+57 = 147 \quad 180-147 = 33$

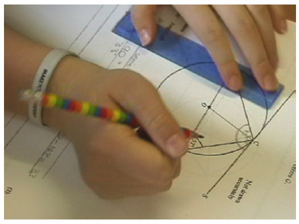
$CAB = 33°$

They found part (b) far more challenging and it is interesting to examine how they eventually arrived at a solution via various blind alleys, prompts from us and occasional flashes of insight.

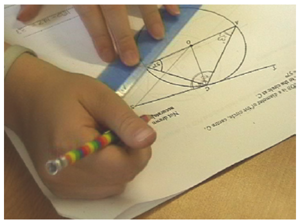
A powerful approach for solving such tasks is to adopt a dual strategy of i. working forwards from the givens and ii. working backwards from what one is required to find. Lorna may have had some intuitive sense of this latter strategy as she spent some time tracing back and forth over an arc denoting the required angle ACS (see below, left). Further, she then traced over the Z-shape SCBO several times and referred to the angles at C and B as ‘alternate angles’ (below, right).

The angle at B is given; so, if this can be used to find the size of the alternate angle at C (ie angle SCB), and with angle ACB known from part (a), it would then be possible to find the required angle ACS. However, Kerrie correctly pointed out that the lines SCT and BOA (ie the tangent and diameter) are not parallel, so the ‘alternate angles’ relationship is of no help. (Angle SCB can be found from the *Alternate segment* theorem - it is equal to the angle CAB found in part (a); this is presumably how part (b) was meant to be answered, but neither pupil knew this theorem.)



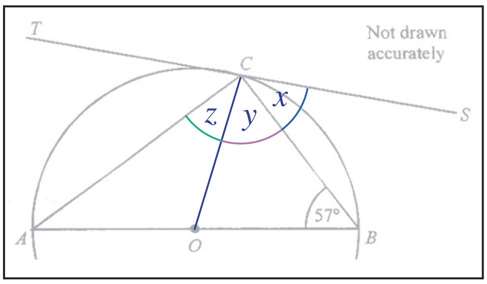
After a while, Lorna drew the radius CO (right), just as she had drawn the radius OD in the earlier question. Again, this was a potentially powerful move, though at this moment Lorna did not seem fully aware of this, since she quickly went on to draw the line OS (below, right).



In contrast to the radius, line OS can not carry any useful information since the position of S is essentially arbitrary (as long as it is somewhere on TC produced) - it is only there to enable us to refer to the tangent. When we asked Lorna why she had drawn OS she merely replied, “Don’t know - to see whether it can help us”.

Such a passive, see-what-emerges approach is a sensible default strategy. However, in this case it did not seem to be getting Lorna and Kerrie any further.

The two pupils had not made any use yet of the fact that the line SCT was a tangent to the circle, even though Lorna had drawn the radius from the centre to the point C where the tangent touches the circle. So we decided to draw their attention to the tangent. The ensuing discussion about the various angles at C (which, to help the reader, have been labelled x, y and z in the diagram below), went as follows:



DEK What is this line here (TCS)?

Lorna (0 min: 02 sec) A tangent.

DEK Right.. What do we know about tangents...?

Lorna (0:12) 90 degrees from the centre, or something...

DEK (0:16) Sort of ... you’re half way there.

Lorna (0:18) I’m not very good with tangents!

DEK (0:21) What a thing to say...! [Pause] (0:28) It turns out that... - doesn’t look like it actually

Kerrie (0:31) - so they are parallel!

DEK (0:35) - those? Oh gosh, don’t know... you’ve got to be careful to... [points to ‘Not drawn accurately’]

Kerrie (0:39) - they don’t look parallel.

DEK (0:44) Well it turns out that there’s a right angle between the tangent and the radius [traces lines with finger]. You happen to have drawn the radius, which I think is a very neat idea... so we’ve got a right angle there [points to x+y].

Lorna (0:58) - right angle here [marks the right angle in the conventional way].

DEK (1:00) So what do we need to get our answer? [ie to find the angle x+y+z in the above diagram] [pause] (1:07) We want this whole angle, right? [traces arc with finger to show x+y+z]

Lorna (1:09) - yes, wouldn’t this have to be 45? [points to angle x] If you split this one in half [points to x+y] you are left with 45 there [x] and 45 there [y].

DEK (1:16) How do you know it is cutting it in half? You’re using your idea of last time [of halving]. Don’t forget this point [C] can be kind of anywhere along here really [traces along arc of circle].

Lorna (1:24) - yeh... [pause]

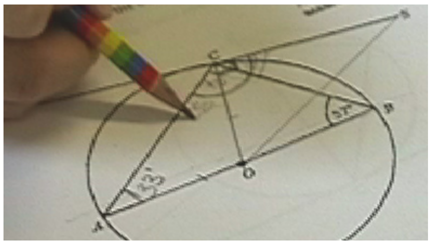
DEK (1:30) If we know this bit’s 90 [points to angle x+y]... and you want to find that whole angle [traces arc with finger to show x+y+z], what’s the bit we don’t know yet? [hoping that she will point to z]

Lorna (1:38) What we don’t know is this bit [points to x].

DEK (1:42) Well yeh, I don’t know that bit [x] but I know that that bit [x] and that bit [y] is 90.

Lorna (1:46) Yeh... [pause]

DEK (1:49) What’s the bit I still don’t know?



Lorna writes “33” (see below)

Lorna (1:53) Well that bit [points to x] and that bit [y] is 90 and that bit [z] and that bit [y] is 90, so that [z] will have to be the same as that bit [x]

DEK (2:01) - oh, right! Yes it would actually. That’s brilliant. [Pause] (2:07) Can we find that bit [points to z] ? [Pause]

Kerrie (2:17) I’m getting lost...

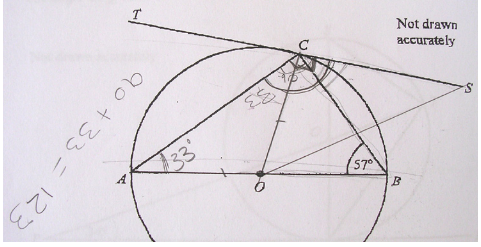
Lorna (2:20) [Turns the question paper to face Kerrie, so she can see more clearly] (2:24) Yeh, because these are both radii [points to AO and OC] ...so this [triangle AOC] has to be - um - isosceles [pause]. (2:33) So this angle is the same as that angle [points to base angles of isosceles triangle AOC] so this angle [at C] must be 33 [writes “33” on diagram (see above)]

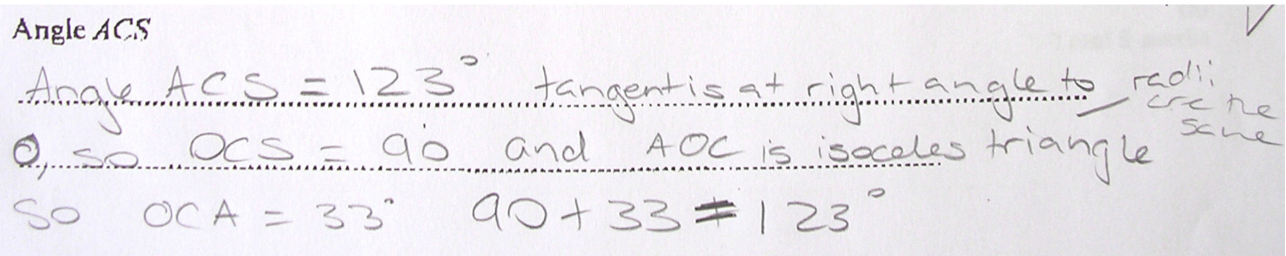
Kerrie (2:38) - ah!

DEK (2:40) Very nice!

Lorna (2:44) And so altogether... [writes at side of diagram: 90 + 33 = 123].

(3:01) [Writes answer in space provided:]





This excerpt is only 3 minutes long, though when one plays it through it seems to be full of uncertainties and lengthy pauses - which we as teachers tend to find very uncomfortable and do our best to fill! However, our interventions seemed to have little immediate impact, indeed they seemed to be getting in the way of the pupils' own thinking. Our aim was to draw out the fact that the angle between the tangent TCS and the radius OC (ie the angle $x+y$) is 90˚ and that all we then need for the required angle ($x+y+z$) is to find the angle z. But as soon as we initiated the discussion about the tangent, Kerrie interjected with "so they are parallel". We were somewhat thrown by this (and did our best to ignore it!) but it suggests that she was still thinking about the idea of alternate angles that we had discussed several minutes earlier. Similarly, as we were trying to focus on the idea that the required angle $x+y+z$ can be partitioned into $x+y$ and z, Lorna seemed to be thinking about the partition $y+z$ (which we knew from part (a)) and x. Then, quite unexpectedly, she brought these two viewpoints together with the sudden insight that if $x+90$ and $90+z$ describe the same angle (namely $x+y+z$), then x must equal z. Astonishingly too, following this insight the remaining steps in the proof seem to have fallen into place instantly for Lorna. Thus, in response to Kerrie's plea for help ("I'm getting lost...") Lorna immediately embarked on an explanation of how to find z.

Thus, although our interventions succeeded eventually (at least with Lorna), they also disrupted the pupils' own thinking. This underlines the importance of giving pupils plenty of time to explore their own ideas - as well as to come to terms with ours.

DISCUSSION

In looking back over these two episodes, it should be made clear that though the pupils were being asked to construct deductive arguments, they were not required to work at a general level and to construct full-blown proofs. Rather, they were given and asked to deduce specific values of angles. In an earlier study (Hoyles and Küchemann, 2004), we found that seeking specific values is substantially less demanding than proving a general relationship and this may be a sensible way of helping pupils make the transition that Dreyfus (1999) talks of here:

> ..the requirement to explain and justify their reasoning requires students to make the difficult transition from a computational view of mathematics to a view that conceives of mathematics as a field of intricately related structures.

The two GCSE questions discussed in this paper were designed to test pupils' knowledge of the circle theorems and their ability to apply them in simple, one- or two-step deductions. The two pupils that we observed did this quite successfully for the first part of each question (where they applied *Angle at the centre* and *Angle in a semicircle*, respectively). However, they seemed to have forgotten the theorems that were meant to be used for the second parts and so for them these items became much more complex (and in turn became much more interesting for us). As the pupils' efforts demonstrated, these items can be solved without the circle theorems, by going back to the properties of the radii of a circle, of isosceles triangles, of the interior angle sum of triangles, etc, ie the properties from which the theorems themselves are conventionally derived. At the same time, their efforts also show just how intricately related the structures are, to use Drefus's words - and how difficult it can be to see potentially relevant properties and connect them into a coherent argument. The two pupils were able to draw out some quite hidden properties (such that if a point is on a circle its distance from the centre is the same as that of other such points from the centre) but had difficulty spotting others (eg that the angle between a tangent and radius is 90˚). They also had difficulty in determining the degree to which the various features of a diagram are constrained, leading them to make assumptions that did not necessarily follow from, or may even have violated, the givens (see eg Dvora and Dreyfus, 2004).

For the pupils, there are several ways in which this work could be taken forward. One step would be to develop greater familiarity with the circle theorems. More interesting, perhaps, would be to help pupils become more aware of the strategies that they and their peers use, and also to explore diagrams that conform to particular givens and consider how different features of the diagrams are constrained.

Acknowledgements

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