Hewitt, D. (Ed.) Proceedings of the British Society for Research into Learning Mathematics (25(3) November 2005
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INTEGER OPERATIONS IN THE PRIMARY SCHOOL: A SEMIOTIC ANALYSIS OF A "FACTUAL GENERALIZATION"

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This study attempts to better understand how children learn integer operations after the 'dice games' approach of Linchevski & Williams, through Radford's semiotic analyses of the means of objectification. In this paper we analyse of pilot study data from an ongoing research. Following a short review of Radford's account of the semiotic processes, it is argued that in the dice games approach factual generalization can be analysed as a three stage semiotic process of reification. Hence, connections between Sfard's theory of reification and Radford's semiotic analyses of the means are made.

INTRODUCTION

The instructional approach of the 'dice games' for integer operations instruction proposed by Linchevski & Williams (1999) was underpinned by the theory of *reification* (Sfard, 1991; Sfard & Linchevski, 1994). The researchers' goal was to achieve the intuitive reification of integers through their addition and subtraction. However, the explanation of the transition of students' concept formation processes from the one stage of the theory of reification to the next is inadequate (Goodson-Espy, 1998). In this research it is being hypothesised that this knowledge gap relates directly to the insufficient investigation of semiotic mediation in the construction of the schema of a new mathematical object.

Semiotic mediation is underinvestigated in relation to the instruction of integers based on the dice games approach of Linchevski & Williams (1999). After a brief discussion of Radford's semiotic analyses of the means of objectification, this paper will focus on a part of the objectification processes taking place in integer instruction. Finally, these analyses will be connected with the theory of reification.

Factual generalisation in algebra

In recent research Radford has tried to address the issue of semiotic mediation in the process he called *objectification,* which is "a process aimed at bringing something in front of someone's attention or view" (Radford, 2002, p. 15). This process takes place through the use of *semiotic means of objectification* (Radford, 2002). Based on these notions, as well as on a new formulation of *schema* (Radford, 2005) which contains a sensual-semiotic dimension, Radford has investigated students' learning of algebra. As a result of these investigations, objectification has been presented as a process taking place through a series of three generalization processes: *factual, contextual* and *symbolic generalization* (Radford, 2003). These three processes, according to Radford (2003), succeed each other so as to produce symbolic mathematical objects. This paper only examines factual generalisation.

> … A factual generalization is a generalization of actions in the form of an operational scheme (in a neo-Piagetian sense). This operational scheme remains bound to the concrete level (e.g., "1 plus 2, 2 plus 3" Episode2, Line1). In addition, this scheme enables the students to tackle virtually any particular case successfully. (Radford, 2003, p. 47)

Radford's (2003) investigation of students' objectification processes was based on a lesson on geometrical sequence generalization. The lesson was based on the following sequence of shapes made up of toothpicks.

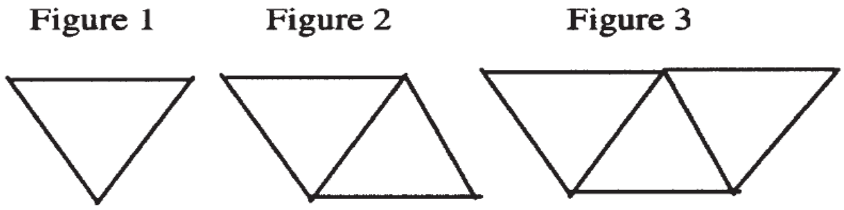

FIGURE 2 The toothpick pattern. (in Radford 2003, p. 45)

In the case of Radford's investigation of students' generalization processes of toothpick patterns, the students abstracted an operational scheme for the calculation of the toothpick number of a figure. The operational scheme was constructed at the concrete/situated level, which in the case of the generalisation of patterns is the numerical level. However, the concept of the general figure was not abstracted. Therefore, no new mathematical object entered the discourse. This scheme allowed the calculation of the toothpicks of any given figure, but the syllogism remained context-bound.

Factual generalisation begins with the manipulation of the above concrete objects through the use of deictic semiotic means (deictic gestures, linguistic terms, rhythm), which enable the students' to abstract an operational scheme according to the task needs. The utilization of deictics is enabled due to face-to-face communication. In the following episode, the students tried to find a way of calculating the number of toothpicks of figure 25.

1. Josh: It's always the next. Look! [and pointing to the figures with the pencil he says the following] 1 plus 2, 2 plus 3 […].
2. Anik: So, 25 plus 26 … .
3. Josh: Wait a minute. Yeah, 3 plus 4 is 7, 4 plus 5…so it's 27 plus 26?
4. Anik: Well, because you always…like…look [and she stretches her arm to point to the figures—see the second picture in Figure 3), 3 plus […] it's 25 plus 26. (Radford, 2003, p. 46-47)

In line 1 "Josh realized that there is a pattern linking the number of the toothpicks and the sum of the rank of two consecutive figures" (p. 47). The students used deictic gestures and linguistic terms to construct a proper operational scheme. In terms of deictic gestures, Josh in line 1 uses "pointing to the figure with his pencil" (p. 46), while introducing the operational scheme. In line 4 Anik "stretches her arm to point to the figures" (p. 47). These deictic means allowed the students to focus their attention to the figures, and thus they objectify them. Regarding deictic linguistic terms, "next" in line 1 "emphasizes the ordered position of objects in the space and shapes a perception relating the number of toothpicks of the next figure to the number of toothpicks in the previous figure" (p. 48). The adverb "always" in the same line allows the "general verbal formulation" (p. 48) of the operational scheme. Finally, in other episodes in Radford (2003), it is shown that rhythm and movement can supplement or even replace the use of gestural and linguistic deictics, as they achieve the goal of indication as well. Summing up through Radford's (2003) words, factual generalization involves:

> …the students' construction of meaning has been grounded in a type of social understanding based on implicit agreements and mutual comprehension that would be impossible in a nonface-to-face interaction. … Naturally, some means of objectification may be powerful enough to reveal the individuals' intentions and to carry them through the course of achieving a certain goal. (p. 50)

THE GENERAL CONTEXT OF THE DICE GAMES

The dice games (Linchevski & Williams, 1999) are four games aiming at the instruction of integer addition and subtraction. They take place in small groups of four children, in which the students are arranged in two teams of two. The teams compete to win the games by collecting points for their teams through throwing dice and recording them on their team's abacus. In the variation of the dice games from which the data of the following section were extracted, the students collect winning and losing points for their teams: a team wins if its score (the difference of its winning and its losing points) is 8 winning points and loses when its score is 8 losing points. The participants were Year 5 students from Greater Manchester.

A FACTUAL GENERALIZATION OF THE COMPENSATION STRATEGY

The episodes below are seen as co-constituting a factual generalization process. The students in the episodes run out of space to add the necessary cubes on the abacus. For the first time the students needed to understand that adding a number of winning points is equivalent to subtracting the same number of losing points or vice versa. This understanding was based on the introduction of a new concept in the discourse, the concept of compensation of winning and losing points.

In episode 1, Frank and Nicole had to add 2 winning points and 1 losing point. They added 1 winning point (yellow cube) and 1 losing point (red cube) and they still needed to add 1 more winning point. As their yellow cubes column was full, they could not add 1 yellow cube as needed.

<u>Episode 1:</u> Minutes 15:10-16:03. ""…" indicates a pause of 3 sec or more, and "." or "," indicate a pause of less than 3 sec" (Radford, 2003, p. 46).

Harry: Well you got 1 there [he points to the yellow cubes], so you take off that 1 [he points to take off a red cube] and that would add 1 there [he points to the yellow cubes], forget about that 1 there [he points to the cube on the table] cause you don't need it, cause that losing point there [he points to the red cubes on the abacus] counts as that winning point.

Researcher: Can you show us what you mean?

Harry: That's a losing point, so you take off the losing point [he takes it off] and that counts as this winning point here [he shows 1 yellow bead which is not on the abacus and cannot be added because there is no more free space]. And so then you've only got 1 winning point [he touches the top yellow cube on the abacus] and you're just taking away a losing point [he points to the losing points] so that…

Researcher: So I guess you are suggesting that instead of putting 1 winning point here [pointing to the top of the yellow cubes on the abacus], because we cannot do that [he shows that it can't fit in]…

Harry: Take off a losing point [he points to the red cubes and the researcher takes 1 off] which counts as that winning point [he points to the yellow cubes] cause it's a losing point…

Harry in the above episode explains (for a number of times) why he proposes taking away 1 losing point instead of adding 1 winning point. Here Harry constructed the process of the compensation of the addition of a winning point with the subtraction of a losing point. Moreover, the idea of retaining the correct difference between winning and losing points has not been expressed explicitly.

<u>Episode 2:</u> Minutes 19:44-20:13. Olga and Harry need to add 4 losing points but only 3 fit on their abacus.

Olga: … just adding on 4 [simultaneously with Harry]. It won't fit [the column only has space for 3 more losing points].

Harry sticks 1 extra red cube on top, but Olga corrects him as follows:

Olga: And if it don't fit, you're gonna take one off [she takes 1 winning point off], so that means you don't…

Harry: It's still…wait, if you just put on that [he puts back the winning point and the extra losing point], how much difference will there be? … So there's 2 difference, so…

Olga: Yeah, you can take that 1 off [she takes off the extra lost point] and that 1 off [she also takes off the 1 winning point], there's still 2 difference.

Harry: …there's still 2 difference [simultaneously with Olga] …

In episode 2 Harry and Olga still need to reconstruct the process of subtracting a losing point instead of adding a winning point, but this time the semiotic process is condensed and there is explicit reference to achieving the correct difference of winning and losing points.

Episode 3: Minutes 21:20-21:30.

Harry: So it's just adding on 1 [he points to the top of the red cubes' column], but there is no space.

Olga: No space, so… so you take off another yellow. [She removes 1 cube from the yellows on their abacus]

Harry: OK. [Harry agrees and does not ask for any further explanation]

In episode 3 there is no longer a need to indicate any process establishing the equivalence of adding 1 losing point and subtracting 1 winning point. There is only indication to the product of the process. Moreover, after this point in the game Harry and Olga only needed to indicate the product of the process without reference to the process itself.

To sum up, the three episodes show how the *compensation* of the addition and subtraction of winning and losing points is introduced:

1. First through a lengthy and detailed semiotic construction of the process and reference to its product.
2. Secondly through a semiotic reference to the process again, this time with an explicit reference to the key idea of constructing the correct difference of winning and losing points, followed by reference to the product of the process.
3. Just the reference to the product is sufficient – the factual generalization is complete.

A BRIEF ANALYSIS OF THE SEMIOTIC MEANS IN THE EPISODES

In episode 1 there is extensive use of deictic gestures in the form of pointing to yellow and red cubes: in every explanation offered by Harry he is pointing or touching or moving cubes. Also, deictic linguistic terms, often combined with deictic gesturing, like "this" and "that" are used. However, the concept of difference of winning and losing points has not been introduced explicitly yet. In episode 2, the use of deictic gestures and deictic linguistic terms is similar to that in episode 1, only this time it is not as extensive as before. The main difference here is the repeated use of the term "difference", which introduces a new abstract situated characteristic to the discourse. Finally, in episode 3 the use of deictics is very limited, as the students now need to focus (and indicate) only on the resulting action.

CONCLUSION AND CONNECTION WITH THE THEORY OF REIFICATION

As discussed above, initially a process is constructed based on deictic semiotic means. In Sfard's (1991) terminology, at this point a new process has been *interiorized*. Next, a new concept enters the discourse through the use of an additional linguistic term: the "difference" of winning and losing points. The introduction of the notion of difference allows the process to become shorter and less deictic activity is necessary. However, indication to the process is still needed. We suggest that in Sfard’s (1991) terminology this is when *condensation* occurs. We recognize the introduction of the notion of “difference” in the process as catalytic for achieving the condensation of the process. Finally, reference to the process is no longer needed. We suggest that the *reification* of adding and subtracting winning and losing points according to the compensation strategy has been completed. We believe these connections improve the understanding of both factual generalization and reification. Though the connection of factual generalization and reification in the instruction of integer operations has not been exhausted, we propose that some first general connections have been accomplished.

REFERENCES

Goodson-Espy, T. (1998). The roles of reification and reflective abstraction in the development of abstract thought: transitions from arithmetic to algebra. *Educational Studies in Mathematics*(36), 219-245.

Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in 'filling' the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics* (39), 131-147.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, *22*(2), 14-23.

Radford, L. (2003). Gestures, speech, and the sprouting of signs. *Mathematical Thinking and Learning*, *5*(1), 37-70.

Radford, L. (2005). The semiotics of the schema. Kant, Piaget, and the Calculator. In M. H. G. Hoffmann, J. Lenhard & F. Seeger (Eds.), *Activity and Sign. Grounding Mathematics Education* (pp. 137-152). New York: Springer.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1-36).

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification: the case of algebra. *Educational Studies in Mathematics*, *26*, 87–124.