MATHEMATICAL ABSTRACTION: A DIALECTICAL VIEW

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In the classical Aristotelian empiricist view, abstraction is associated with an ascending developmental process from the concrete to the abstract. In this paper, however, I argue that the development coming about in the formation of mathematical abstraction can be best portrayed as a dialectical development to and fro between the concrete and abstract. This argument is exemplified on the basis of the verbal protocols of two students working together on a task connected to sketching the graphs of absolute value linear functions.

Abstraction is a term often linked with an empiricist philosophy tracing from the writings of, for example, Aristotle to Locke to Russell. In this tradition abstraction is considered as higher-order knowledge which consists of ‘classifications’ (Dienes, 1963) and ‘generalisations’ (Dreyfus, 1991) arising from the recognition of commonalities isolated in a large number of specific instances (see Ohlsson and Lehtinen (1997) for a critique). This tradition sets the concrete and abstract as bipolar opposites and views abstraction as an individual process which is a transition and ascending development of mathematical concepts from the concrete to the abstract (see Noss and Hoyles, 1996 for more on this).

However, in recent years, many expressed dissatisfaction with the consideration of abstraction as an ascending development from the concrete to the abstract (e.g., Noss and Hoyles, 1996; van Oers, 2001). For example, Hershkowitz, Schwarz and Dreyfus (2001) have recently developed a dialectical materialist approach in which, by drawing on Davydov’s (1990) ‘method of ascent’, these authors view the genesis of an abstraction as an undeveloped initial entity which develops through the use of mediational means and social interaction. This development is not from the concrete to the abstract but, rather, a dialectical *to and fro* between the concrete and the abstract. As a result of this development, they argue, the students construct and hence are enriched with an abstract knowledge structure by reorganising previously constructed mathematical knowledge into a new mathematical structure. However, Hershkowitz et al., in my opinion, do not clearly state their view on the concrete and abstract; further to this I also feel that the dialectical relationship is not made sufficiently explicit in their analyses and discussions. It is my intention in this paper to delve into the dialectical relationship between the concrete and abstract in the course of formation of abstraction and explicate the nature of this dialectic on the basis of student verbal data.

Before doing this, however, it is essential to clarify my intention with the terms ‘concrete’ and ‘abstract’. On the basis of reviewing the relevant literature (e.g., Davydov, 1990; Ohlsson and Lehtinen, 1997; Noss and Hoyles, 1996), my position on the abstract and concrete is as follows: the abstract is complex, complexity in terms of the depth of an idea in the sense of having some “other ideas as parts”

(Ohlsson and Lehtinen, 1997, p.42), goes beyond particular instances and is related to theoretical thought in the sense of Davydov. The concrete on the other hand is concerned with particular instances and experiences and is often related to empirical thought in the sense of Davydov. In Davydov's (1990) view, empirical thought is concerned with establishment of "particular connections and relationships" (p.253) which can be expressed verbally "as the results of sensory observations" (p.255) (e.g., observing similarities and differences between things). However, establishment of internal links or "essential relationships [which] cannot be observed directly by the senses, since they are not given in available, established, resultative, and disassociated being" (p.255) requires theoretical thought which he describes as "an idealisation of the basic aspect of practical activity involving objects and of the reproduction in that activity of the universal forms of things, their measures, and their laws" (p.249).

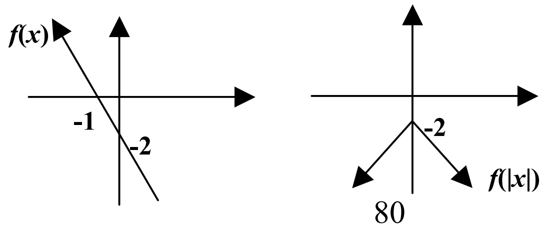
BACKGROUND: THE STUDENTS AND TASKS

To exemplify the dialectical relationship between the concrete and abstract, I report on the verbal protocols of two 17-year-old girls (H&S) who worked together, with an interviewer assisting them, on four sequential tasks connected to sketching the graphs of absolute values of linear functions. These two students were selected for this study via a diagnostic test which was prepared to ensure that the students had not encountered the content of the tasks but had sufficient knowledge to embark on the tasks.

In this paper, I present H&S's verbal protocols generated through their work on the second task, by the end of which the students were expected to construct a method to sketch the graph of $y=f(|x|)$, given the linear graph of $f(x)$. This task was composed of five questions. The first question asked the students to draw the graph of $f(|x|)=|x|-4$ and to comment on any patterns in the graph. In the second question, students were asked to report on the relationships between the graph of $f(|x|)=|x|-4$ and the graph of $f(x)=x-4$. In the third question, the students were asked to sketch the graph of $f(|x|)=|x|+3$ by using the given graph of $f(x)=x+3$ as an aid. In the fourth question, students were presented with four linear graphs without equations (referred to as 4A-4D in the paper) and were asked to sketch the graph of $f(|x|)$ for each one of these graphs. The final question asked students to explain a method which can be used to draw the graph of $f(|x|)$ by utilising the graph of $f(x)$.

VERBAL DATA

The students (H&S) answered first three questions by sketching the target graphs by substitution and making some comparative comments between the graphs of $f(x)$ and $f(|x|)$. They later moved on to question 4 and sketched the graph of $f(|x|)$ for 4-A first by finding its equation and then by substitution. They obtained the graph below.



Following this, S realised a symmetrical relationship in the graph of $f(|x|)$.

60S:	Look, for example, it is the same in this graph as well. I mean in this graph the part of $f(x)$ until the y-axis remains the same and then the remaining part is taken symmetry
61H:	That means the symmetry is starting from the value of y at x=0? Are you saying that the graph remains the same until x=0?
62S:	Yes. For example, the graph takes the value of -2 at x=0 and the graph is left unchanged until the point of (0,-2). But after that a symmetry is taken… I don't know how to say it…
64H:	Look now… as from the point of (0,-2), the remaining part is taken symmetry in the line of y=-2

H&S later moved on to question 4-B and attempted to apply their observations.

68S:	Let's first draw the graph [for the graph of $f(x)$ given in question 4-B] by considering what we've just found out and then control it by substituting, right?
69H:	[They turn to the graph given in 4-B]. So, according to our findings, the graph of $f(x)$ will be the same until the point at x=0.
70S:	Uh huh, yeah it is the same
71H:	After that what?
72S:	For the rest, the symmetry…
73H:	We will take the symmetry…

H&S applied their observations to draw the expected graph in question 4-B correctly. But they obtained an inaccurate graph of $f(|x|)$ for question 4-C and S realised that there was something wrong with this graph (NB: 'I' refers to the interviewer).

100S:	Look, it should have been towards here not there!
101H:	Why should it have been so? It is not necessary… look, these two rays [in the graph of $f(|x|)$] should be symmetric in the line of y=2
102S:	OK, look isn't this line of y=2?
103H:	Yes
104S:	If we take the symmetry of that part in the line of y=2
105H:	Which part?
106S:	That part.
107H:	No! We shouldn't take the symmetry of this part
108I:	S says that symmetry of the ray in the left side of the y-axis should be taken according to the line of y=2
109S:	Yes, I mean shouldn't we take the symmetry of this part? Look at this graph [see the graph given above] the part of $f(x)$ until the y-axis is unchanged and the remaining part is reflected.

The students' arguments continued for a while but they could not convince one another as to the (in)accuracy of the graph. To assist the students, the interviewer intervened and asked them to look into all graphs of $f(|x|)$ and decide which part of $f(x)$ remained the same and which part changed. He also drew the students' attention that on the right side of the y-axis the part of $f(x)$ corresponded to the positive values of x.

123S:	Oh, yes… they are positive and so we don't change them…
124H:	Positive values don't change?
125S:	I mean… we are drawing the graph of $f(\|x\|)$, right? I mean we are talking about these graphs…
126H:	Yes, so?
127S:	Umm… the absolute value sign is always outside of x, I mean it is $\|x\|$
128H:	Positive values don't change in the absolute value sign…
129S:	Exactly, it changes the negative values

H&S also established that the part of $f(x)$ at $x<0$ needed to change, arguing that absolute value changes negative values and hence the graph of $f(x)$ at $x<0$ must be different from the graph of $f(|x|)$.

136S:	Because positive values of x remain unchanged in the absolute value sign, but negative values of x must be multiplied by minus to go out of the absolute values sign… thus we can say that whatever changes occur in the graph of $f(x)$, it must be at the negative values of x
137H:	OK, look let's put things together: as far as I understand, the part of $f(x)$ on the right hand side (of the y-axis) should remain the same, no matter what. And the part on the left hand side (of the y-axis) should change...

As answering question 5, the students explained their method as to how to obtain the graph of $f(|x|)$ by considering one of the graphs of $f(|x|)$:

197H:	We can obtain the graph by taking the symmetry in this line [of y=-2] and this line always crosses the y-axis… I mean … when x=0.
198S:	Look, we can obtain the graph [of $f(\|x\|)$] by first leaving the right part [of $f(x)$ at the positive values of x] as it is
199H:	Right! And also we need to take the symmetry
200S:	We also take the symmetry of the part I mean left part [of $f(x)$ at the negative values] in the line…in the line… oh! It's difficult to say now…
204H:	Look, we can say that the line always crosses the y-axis and the crossing point is determined by substituting x with 0 in the equation…

DISCUSSION

It is clear from the students' verbal data that H&S constructed a method to view the graphs of $f(|x|)$, that is, reflecting the part of $f(x)$, for $x<0$, in the horizontal line through the y-intercept of $f(x)$ but keeping the part of $f(x)$, for $x>0$, unchanged. I call this 'reflecting' method, which was an abstract structure. It is clearly a complex structure in the sense of having knowledge components as parts such as the notion of symmetry, reflecting, features of absolute value and properties of linear functions and Cartesian grids. This complexity is related to the fact that in their 'reflecting' method, H&S had to integrate these components into a single structure and needed to employ theoretical thought (see below) to develop interconnections amongst these components. Further to this, this method was not concerned solely with a particular graph of $f(|x|)$; this is clear from the students utterances such as 136S and 137H.

However, construction of this method was achieved through particular instances and (concrete) examples which gave rise to the initial form of the 'reflecting' method (60S-64H). When H&S came up with their initial 'reflecting' method (e.g. 60S), they were drawing heavily on the specific features of the graph of $f(|x|)$ at hand and were observing the similarities between the graphs of $f(|x|)$ that they had obtained earlier. For example, they focused on specific points from which symmetry starts (e.g., "from the value of y at x=0" – 61H; "the graph takes the value of –2 at x=0" – 62S; "the point of (0,-2) – 64H); they reported certain similarities between the graphs of $f(|x|)$ (e.g., "it [the symmetry] is the same in this graph as well" – 60S).

This 'reflecting' method proposed in 60S was not applied consistently to draw the graphs of $f(|x|)$ by the students precisely because it was, as Davydov (1990) suggests, in its immature first form which was vaguely developed, the vagueness related to under-specification as to which part of $f(x)$ was reflected in the line through the y-intercept and the reasons for this. For example, when H&S applied this method to draw the graph given in question 4-B, they were successful despite all the vagueness of the method. Yet when they applied it to the graph given in question 4-C, they obtained an erroneous graph. Even when they applied this method with some success in question 4-B, H&S were still concerned with specific features and properties of the graphs; e.g., "the graph of $f(x)$ will be the same until the point at x=0" – 69H. This clearly suggests that in the course of their construction of the 'reflecting' method, H&S were concerned with particular instances and employed empirical thought.

Although H&S came up with the initial 'reflecting' method through empirical thought, this was not sufficient for them to achieve the construction of this method which required theoretical thought. This observation could be best illustrated on the basis of H&S's work between the utterances 100S-109S and 123S-137H (not all shown). In the former interval, H&S were talking about the accuracy of their erroneous graph of $f(|x|)$ which they sketched by using their initial 'reflecting' method. S asserted that the graph was wrong and tried to convince her partner, H. However, neither of them produced convincing arguments as to the (in)accuracy of the graph which remained unresolved. However, in 123S-137H, with the assistance of the interviewer, they constructed the 'reflecting' method. The telling difference between the students' arguments in these two intervals is related to the mode of reasoning that they employed. In Davydovian terms, H&S can be said to have employed empirical thought in 100S-109S but theoretical thought in 123S-137H. To clarify this, I focus on two example explanations from S (109 and 136); In both of these statements, S was arguing that the segment of $f(x)$ at x<0 should be reflected when it is transformed into the graph of $f(|x|)$.

S's first statement (109S) was basically concerned with justifying the argument of "taking symmetry" on the basis of her observation of ostensible features of a previously drawn graph. In other words, she was interconnecting features of 'reality', i.e. suggesting reflection of $f(x)$ at $x<0$ due to the similar features of the earlier graph of $f(|x|)$. However these observations as such did not lead the students to further clarify their 'reflecting' method. In contrast, S's statement in 136 was concerned with highly developed mathematical reasoning. She was reproducing "universal forms of things, their measures, and their laws" (Davydov, 1990, p. 249), e.g., "positive values of x remains unchanged in the absolute value", "negative values of x must be multiplied by minus." In this connection, she was, at this point in her development, establishing interconnections amongst the structural features of absolute value, the notion of symmetry (she expressed this as "changes"), a linear graph of $f(x)$ and positive/negative x-axis sub-domains. From Davydov's perspective, these interconnections are "internal, essential relationships" which are not immediately available to the "sensory observations" (ibid., p.255). However, it is interesting to note that, when asked to explain their method, one can see the students returning to again particular instances and properties of the graphs that they had at hand at that moment: e.g. "taking the symmetry in *this line* … when x=0" – 197H.

These considerations simply suggest a dialectical development *to and fro* between the concrete and abstract in the course of abstraction. An important implication of this dialectical relationship is that both the concrete and abstract are linked rather than detached and they both are necessary for individuals to appreciate the other. That is, the abstract and concrete mutually constitute the 'meaning' and both kinds of knowledge act as resources of 'meaning-making' for each other.

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