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WHAT HELPS STUDENTS WRITE FORMAL PROOFS?

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Undergraduate students starting to work on mathematical proofs face the two-fold challenge of finding words that express reasoning, and evaluating their reasoning against formal mathematical models. Mathematics is simultaneously opened up by opportunities of language, and closed down by logical and conventional practices of formal mathematical discourse. In this paper I examine undergraduates' early written number theory proofs to explore linguistic features that characterise a transitional undergraduate discourse. I present two examples of students' use of language - predictive transformation of argument and use of an ambiguous structural descriptor - and discuss how these serve to structure a deductive approach.

INTRODUCTION

Early undergraduate experience can be seen as an induction into the discourse of formal mathematics, and especially that of proof. At the same time the practices, register and contextual referents that form the discourse are strengthened and reaffirmed by the students' participation (Recio & Godino,2001). There has been much discussion of the conflict between the explicit conventions surrounding formal proof and its role - to explain - in school mathematics (Hanna, 1991). This conflict also arises in undergraduate mathematics when educators need to give formative feedback on students' proofs while upholding the formal model of logical absolutism. This raises questions such as: Can a proof be right but expressed badly? Is it possible to recognise progression in the expression of logic? Can we distinguish language from reasoning? There is little discussion, even in the school mathematics community, of how to improve students' verbal reasoning - perhaps because of the difficulty of getting them to do any at all.

Students find it difficult to write mathematical proofs, in the sense of general arguments based on deductive logic. They often adopt proof schemes that are empirical, even when they judge that such proofs are not appropriate (Harel & Sowder, 1998, Healy & Hoyles, 2000). This suggests that investigating the interaction between the text a student produces and the purposes that the student is trying to achieve may help to understand students' participation in proof. My study therefore takes a discursive approach to the proof texts of early undergraduates. I use the definition of *discourse* as language-in-use, having regularities that may be inferred and described but which are not explicit or defining (Yule and Brown). I compare features of these texts with the discursive features of formal proof texts (Balacheff, 1987, Morgan, 1998), and offer a linguistic perspective on one common student proof scheme, transformational proof (Harel & Sowder , 1998).

DISCURSIVE FEATURES OF FORMAL AND INFORMAL PROOF

I follow Morgan (1998) in describing the linguistic features of mathematical discourses in terms of their ideational, interpersonal and textual functions. The ideational function is the way in which language expresses experiences deemed mathematical. Ideational characteristics of formal texts are that they include specialist words, signs, and grammatical structures such as nominalized verbs (e.g. *conjecture*) by which a process becomes a noun (Halliday & Martin, 1993). Informal classroom discourse instead makes use of subtechnical metaphor (Cameron, 2001), for example, *goes into* for the technical *divisor*. Children's spoken mathematics also makes use of deictic pronouns such as '*it*' to refer to general concepts without having to delineate them (Rowland, 2000). Formal texts are depersonalised, decontextualised and detemporalised, thereby distancing readers from mathematical activity and actors, and suggesting general truths by minimising contextual factors (Balacheff, 1987). This is evident in the high proportion of verbs which convey relational or mental processes, i.e. concerning attributing/ identifying or sensing / thinking, as opposed to material processes concerning doing (Morgan, 1998, p 80).

Portraying mathematical experience as remote from human activity also bears on the interpersonal stance of formal texts i.e. the expression of social relations in the text. In formal texts the authors are positioned as authorities, thinkers rather than doers, and not engaged in debate with readers. They include few uses of personal pronouns and only occasional use of verbs with human actors. The textual function is how the text expresses its communicative purpose, seen in its organisation on the page, in paragraphs, and in sentence structures. e.g. by organising sentences in the form *As* …, *hence* … to demonstrate reasoning. A useful indicator of textual function is the themes of sentences, found at their beginnings in English. In formal mathematics these are often expressions of causality, and this privileges deductive argument over narrative or description of context.

The research questions that structured the analysis were:

1. How do students' texts differ in their ideational, interpersonal and textual functions from formal mathematics texts?
2. Are there any characteristic regularities of an undergraduate discourse? If so, what is the nature of their use within the argument?

METHOD

The data for this study took the form of texts written by students in the first assignment for their undergraduate-level mathematics and education course at a British university. The 21 students were intending teachers, studying solely mathematics for this first year; 15 were women, 6 men; 14 were undergraduates aged around 18-20, with the other 7 being older post-graduate students 'converting' to mathematics after taking degrees in other areas. The students were following a Number Theory syllabus consistent with first year mathematics majors. They were given the task as part of the number theory course, to be completed individually at home. The task (Fig 1.) is one that had been developed and refined over several years by the teaching team, with a pedagogic intention of modelling and encouraging written reasoning about elementary number theory concepts.

Fig. 1 The Task

For any given positive integer a, denote by $N(a)$ the number of positive divisors d of a such that d and a/d are coprime.

For example if $a = 12$

d	1	2	3	4	6	12	
a/d	12	6	4	3	2	1	
coprime?	y	n	y	y	n	y	so $N(12) = 4$.

Check that you understand the definition of $N(a)$ by working out $N(50) = 4$ and $N(84) = 8$. Comment on the truth of the following statements:

"If $N(a) = 2$ then a is a prime number"

"$N(a)$ can be any even number"

"If a is the product of two primes then $N(a) = 4$"

The students were told they would receive detailed feedback on accuracy, reasoning and writing style. The responses can thus be seen as students' first public attempts during the course to engage in a formal written discourse.

Analysis

The analysis consisted of two parts: a product-level collection and categorisation of linguistic items identified *a priori* as significant in a proof discourse, and a process-level - richer and deeper - analysis of how these contribute to the students' claims towards formality and proof. For the product level analysis, a framework of linguistic features was associated with the formal register, or with the expression of empirical/ deductive reasoning. This provided the basis of the first approach to the transcripts: a systematic and to some extent quantitative search abstracting and classifying linguistic features recognised to be of interest: themes, types of process, modified verbs, nominalized verbs, personal pronouns, deictic pronouns, repeated phrases, and patterns of sentence construction.

The second approach to the transcripts was to return to the students' individual texts to examine the linguistic features identified *in situ*. This analysis is dynamic, looking at how the items are combined on the different timescales of phrase, sentence and paragraph level (Cameron, 2001), with the aim of examining the effect in forming the student's textual argument, in positioning the student interpersonally, and within formal and informal discourses of proving.

FINDINGS AND DISCUSSION

Is mathematics concerned with activity or reasoning?

Research in Number Theory (Zazkis & Campbell, 1996) has described students' preference for performing material procedures as a problem solving activity. The table below shows that this was not dominant as an organisational strand in these texts. Just under half of the students' themes concerned argument, serving both textual and ideational functions of establishing the text as concerning reasoning about logical statements.

Categories of theme	**Number of instances (total = 473)**
Reasoning (e.g. *If, Therefore, As, So*)	217
Question heading/ restatement	68
Object (e.g. *N(a)*)	63
Nominalized verbal or mental process (e.g. *The statement…*)	57
Deictics (*It, This)*	29
Verbs (e.g. *Looking at*)	29

The texts as a whole show a similarly restricted use of material processes as verbs. Only some 60 different verbs were used (in 687 occurrences); with a huge preponderance of forms of *to be* or *to have* i.e. processes describing relations between objects. The non-relational verbs were usually in the passive tense, and any actors tended to be mathematical objects rather than humans e.g. *the statement applies* ... Where personal pronouns were used they were usually associated with the classroom metaphor "saying/seeing = understanding", and often in the modal form *we can say*. Thus, the full analysis of ideational, interpersonal and textual functions suggested that most students were writing within a recognisably formal mathematical genre, having adopted the characteristics of reasoning themes, downplaying of human actions, claims to authority and prevalence of relational over material processes.

Types of process used

In the previous section I described the few instances of material processes as verbs. Perhaps unsurprisingly, these occurred when students described more complex reasoning. For example, while almost all students use the relational verb *is* for the familiar fact that:

> A prime **is** only divisible by 1 and itself;

they use the material processes *come*, *occur,* and *arrive* for divisors of a general integer. Other material processes (*obtained, produced, given, formed)* are used with the symbol N(a) as their object. In these cases N(a) is produced passively, i.e. with no actor, or with the subject being an *It* or *This* that refers to a worked example placed nearby the text. N(a) is thus being described as the product of a material or mental process rather than as an object which is related to *a*. The task text does suggest a function interpretation by using the notation N(a) but does not explicitly define a function N. The students do adopt the language of N(a) as an outcome, but only make deictic references to the function itself.

Relational process verbs such as *be* and *have* are seen as formal in that they distance proofs from activity, but when the students use them in a modal form *will be* they can also suggest the carrying out of procedures. Thus Jenny writes:

> True. If a number a is not a square number then its divisors form pairs, which **will either be** coprime or not, ∴N(a) will be an even number. If a is a square number, it **will have** pairs of divisors, and its square root as a divisor. N(a) **will still be** an even number because a/d where d is √a **will not be** coprime &∴**does not** change N(a). (my emphasis in bold)

The use of the future *will* instead of the present tense of mathematical propositions, or the conditional *would* of a hypothetical if-then formulation, suggests a potential action i.e. a generalisation beyond specific instances. Jenny's echoing of *d* and *a/d* from the layout in the question also suggests that her reasoning includes a mental carrying out of that process. Therefore she is not reasoning solely from logic applied to relations, properties and definitions although that is the tenor of her argument as shown by her themes.

In the set of texts 115 of the 687 verbs used were similarly modified uses of *will be* or *will have*, suggesting the outcome of a potential unspecified action of discovery, perception or construction. Uses of a straightforward future tense related to a specific time-related event were unusual and different in their nature, e.g. *I will test one more*. Thus, although material processes were downplayed in the themes of the texts, they are retained as potential actions when elaborating the generality of the argument. The evidence is that material processes are not completely abandoned as in formal mathematical discourse but referred to indirectly. Students expect the reader to understand that they should imagine the outcome of carrying out processes.

Jenny's proof (above) is an example of what Harel and Sowder (1998) describe as a *transformational* proof scheme common amongst college students. Such proofs are characteristically goal oriented; they consider aspects of generality of the conjecture; they involve a transformation of an image; and anticipation of its outcome acts as a structure for deductive inferences that will yield an argument. Jenny considers aspects of generality: a square or not. As she changes a to be a square – the transformation - she considers what effect this will have on N(a), and by anticipating the effect on the outcome of the counting procedure concludes that it *does not change N(a).* So a second purpose of the *will be* formulation is to signal the transformations that change her reasoning to cover all cases. In her conclusion she has focussed on the important characteristic, parity, rather than value. If you actually did change a from a non-square to a square number, then N(a) would be almost certain to change, and again this shows the potential nature of her action and the fact that she is using it to structure an argument rather than actually performing it on an example.

Jenny refers to the properties of *having* divisors and *being* coprime, with calculation processes implicit. For students writing in this way, the process outcomes could be abstract mathematical objects. A different approach used by many students was to reason in terms of the visual outcome of the worked example, thereby implying a material process and using properties of its representation as a metaphor for properties of the abstract mental object. Many visual terms were used by students e.g., *middle*, *appearing* , with others that could refer to both procedural and visual approaches, e.g. *pairs, sequence, arranging, permutations*. For example Fiona's proof not only uses *grid, table* and *symmetry* but introduces symmetry in an incorrect use of the | symbol meaning "divides into" :

> For any even or odd number, (a), then there **will always be** an even number of divisors. …] Because if b| c then c| b – so all divisors arrive in the grid in pairs. This is shown in the line of symmetry in the table …

In summary there are two prevalent metaphors of potential action, one based on the material/mental process of finding and checking divisors, and the other on the material process of creating the visual outcome. Their use of these metaphors in the text is associated with reasoning. Fiona's is a typical example in that words that carry the metaphor are closely combined with her reasoning words (even *shown*). In the next section I shall look in detail at how one particular word – pair – acted as a *structural descriptor* for the problem, referring to general concepts through repeated and deictic reference (Radford, 2001).

Pairs: a structural descriptor

The word *pair* is not used in the question text, but nevertheless 18 out of the 21 students introduced it to describe the divisors in their reasoning about N(a). The word itself just happens to suit this task; my aim is to illustrate how it is used to structure an argument:

> An even number of divisors can be obtained when a is not square […]
>
> | | | | | |
> |---|---|---|---|---|
> | d | c_1 | c_2 | … | c_n |
> | a/d | c_n | c_{n-1} | … | c_1 |
>
> **Pair** of values which when multiplied together produce *a*. Each **pair** occurs twice (2 permutations), with both elements appearing once as a value in d; and once as a value of a/d. When a value of d, x is coprime to a value of a/d, y, the reverse is also true: a value of d, y is coprime to the value a/d x. Thus N(a) can be an even number.
> An odd number of divisors can be obtained when a is a square.
> In this case, the middle **pair** of values (When d=√a) **will be** identical, meaning they are not coprime, and a second permutation cannot be found. Ultimately only the twin **pairs** are left and thus, N(a) can again be even.
> It therefore follows that for all values of a, where a is a positive integer, will give an even value of N(a).

Tim's ungrammatical first phrase starts, I guess, as a comment on his diagram above but also serves implicitly to define his use of *pair* and *values*. He notices and describes his pairs in process terms, with elements appearing as values of d and a/d. For Tim, values is a secondary structural descriptor: he previously used the notation c_i but then abandoned it. Visually, the variables d and a/d act as row headings with the values as entries. Exactly the same word pattern, *value of d*, *value of a/d* is then used for the coprime argument, and leads to his conclusion for even numbers. Thus, in this first case, *pair* introduces a word pattern that gives the whole argument continuity.

Tim also uses the transformational proof strategy of predicting how the arrangement will change for an odd number of divisors. He now treats pairs as objects and focuses on their properties (middle, twin) and what happens to them. *Ultimately* is usefully ambiguous in signalling both the end of the processes of calculation and transformation, and of Tim's argument. In this penultimate sentence, the nature of the pairs themselves, how they were derived, and what is meant by *twin*, is considered shared knowledge and used as a premise from which to conclude that N(a) is even. As a noun, *pairs* therefore serves a deductive function in encapsulating all that has gone before, and referring back to it when used to introduce the next conclusion

Some part of the power of the word is perhaps in its vagueness. It is possible to move easily between levels of abstraction, for example in saying that *the middle pair of values will be identical* he ascribes a property of *values* to the *pair* as an object. Tim carefully distinguishes two permutations of vertically arranged pairs, d with a/d, and then coins *twin pairs* to describe the effect of horizontal pairing, or pairing of pairs. Other students using *pair* did not attempt this last distinction (Jenny's proof, for example) but their proofs read as equally valid. I conjecture that this ambiguous usage is concise and effective in allowing the reader to interpret the word in any meaningful way, while highlighting the notions of matching and parity that are most relevant to the desired conclusion.

CONCLUSIONS AND IMPLICATIONS FOR TEACHING

The aim of this study was a description of elements of the undergraduate student proof discourse, tracing their relation to everyday, school and formal mathematical discourses. Many of the practices of formal mathematical discourse had been adopted by these students, in particular the depersonalisation and detemporalisation of text and thematising of deductive reasoning. There were also examples of using narrative verbs and empirical justifications more appropriate to the school mathematics discourse. Most of the invalid proofs were marked by reliance on these informal reasoning claims as well as the use of incorrect definitions.

Most of these students adopted a proof scheme which could be termed transformational (in the classification of Harel and Sowder, 1998), in which they started from a typical case with a known process outcome and predicted what would happen if the case changed. These proofs were phrased mostly in terms of potential actions. The phrases *will be* or *will have* signal not only the decontextualisation and detemporalisation of material or mental processes which derive N(a) from a but the generality of the students' reasoning when it is transferred from one case to another.

Students used both material process and visual outcome metaphors to describe properties of N(a), and some of these phrases could be identified as structural descriptors, appearing repeatedly in extended arguments. The most common structural descriptor was *pairs*. I suggest that a key reason for the students' success in using the term is its ambiguity, with both process and visual referents, and the potential to blur the distinction between pairs and the objects that are paired. This structural descriptor acts similarly to the use of nominalization of formal mathematics, i.e. by referring back to previous statements, and presenting them as unarguable. Further research is needed to verify this hypothesis in other contexts, and whether the practice is present in informal proof-making activities that lead on to formal mathematics.

Finally, the finding that students use proofs relying on transforming between cases, suggests it as a proof heuristic that can be integrated with proof by generic examples: think of the typical case. What could go wrong? How can it be resolved? This overall textual function of critiquing an existing argument to some extent reflects the approach of the task, i.e. to set up statements and then comment on their truth, but it may also be a wider feature of students' transitional discourse. In this context a teacher could consider making more explicit use of the *For all* quantifier. It might be useful to reformulate a case-by case proof into a *For all* proof, and this would give students experience of setting up a single notation such as $a = p^m q^n r^o ...$ that offers control of the desired variation of prime factors.

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