Prodromou, T. and Pratt, D. (2005), 'Harnessing the causal to illuminate the stochastic', in D. Hewitt and A. Noyes (Eds), *Proceedings of the sixth British Congress of Mathematics Education* held at the University of Warwick, pp. 136-143. Available from www.bsrlm.org.uk.

HARNESSING THE CAUSAL TO ILLUMINATE THE STOCHASTIC

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This study builds on prior work, which identified that students of age 11 years had sound intuitions for short-term randomness but had few tools for articulating patterns in longer-term randomness. This previous work did however identify the construction of new causal meanings for distribution when they interacted with a computer-based microworld. Through a design research methodology, we are building new microworlds that aspire to capture how students might use knowledge about the deterministic to explain probability distribution as an emergent phenomenon. In this paper, we report on some insights gained from early iterations and show how we have embodied these ideas into a new microworld, not yet tested with students.

EMERGENT PHENOMENA

In the service of making sense of the world, people appear to have an intrinsic desire to attribute meaning to what they observe, a search which leads, in turn, to organisation, the formation of patterns, the encoding of pictures, and simplification. Even complex dynamic systems are simplified into emergent phenomena. When numerous *micro* agents of a system dynamically interact in multiple ways, following local rules and utterly oblivious of any higher-level instructions, they can form higher-level patterns (Johnson, 2001). These kinds of discernible *macro* behaviour are called *emergent phenomena*. An understanding of emergent phenomena is increasingly becoming a core part of scientific knowledge and the adoption of this new perspective is essential to comprehend the world (Wilensky and Resnick, 1999). Indeed, our research aim is to design a microworld setting in which we can observe students harnessing their deterministic causally-based thinking to imbue meaning to probability distribution as an emergent phenomenon.

Based on Wilensky's work (1997) about connected mathematics and emergent phenomena, we defined probability distribution, both discrete and continuous, as a dynamic and complex construct with a coherent personality (Prodromou, 2004). This personality self organizes out of many individual decisions (data) and a global order emerges out of uncoordinated local interactions over its duration. A pattern (probability distribution) emerges out of the anarchy of randomness.

Resnick (1991) has noted a tendency to adopt a centralised mindset when interpreting emergent phenomena. We see a resonance between this tendency and Piaget's seminal work (1975, translated from original in 1951) that reported how the organism fails in the first place to apply operational thinking to the task of constructing meaning for random phenomena. Only much later do we, according to Piaget, operationalise randomness through the invention of probability. In this sense, Piaget offers us a first hint that we only begin to gain some mastery over the stochastic when we learn how to exploit our well-established appreciation of the deterministic. Pratt (2000) reported that students age 11 years articulated meanings for short-term randomness that were pretty well consistent with those of experts. They were able to discuss randomness in terms of unpredictability, lack of control and fairness in much the same way as statistically aware adults might. Nevertheless, they had little or no language for discussing distribution or the *Law of Large Numbers*. In other words, their appreciation of the patterns that emerge in the longer term was not well developed. As Pratt's students worked with a domain of stochastic abstraction, Chance-Maker, they began to articulate new meanings for the longer term. These meanings were causally-based and situated. The study described in this paper builds on those ideas as it attempts to clarify how students at one level let go of the deterministic whilst at the same time re-apply such ideas in new ways to construct probability distribution as an emergent phenomenon.

Much depends of course on the design of the microworld, which must somehow offer us as researchers a *window* (Noss and Hoyles, 1996) on the evolution of such thinking. The microworlds must capture the student's thinking process, or at least a meaningful element of it, by providing sufficient perturbation that we can observe as *thinking-in-change*.

APPROACH

The approach of the current ongoing study falls into the category of *design experiments* (Cobb et al, 2003), a methodology that is sensitising us to the complex *learning ecology* through iterative design of the microworld. As we work with students using successive iterations of the software, we are beginning to recognise the emergence of patterns in students' reasoning.

The process of developing the model had made us particularly aware of one aspect, the aspect of *levels* (Wilensky and Resnick, 1999). We conjectured that, if participants encounter emergence as letting go from determinism, they might think about distributions as complex adaptive systems and somehow harness meanings for the deterministic in making sense of emergent behaviour. The challenge was, and remains, one of formulating a microworld design that embodies at each stage testable conjectures, based on our sensitised appreciation of students' meanings as they shift between the micro and macro levels.

We therefore presented the two levels, micro and macro, in separate projects. In the first project, we fore-grounded causality and in the second project we placed emphasis on emergent distribution. The students first experimented with the micro level project but even at that level they were encouraged to let go of determinism through the introduction of error in the determining variables. In the spirit of *constructionism* (Harel and Papert, 1991), we searched for a meaningful context in which the students would be able to articulate meanings at these two levels. We chose a basketball setting where throwing a ball into a basket emphasised deterministic control but allowed the introduction of error to afford the consideration of the stochastic.

Micro level project

In this paper, we wish to focus on insights gained during use of the macro level project and so we will deal rather superficially with the micro. At the micro level, the student was challenged to throw an on-screen ball into the basket. The student can alter the basket size as well as the speed and direction of the throw.

The task directed attention of the student to causality (speed and angle of throw). Then the student was encouraged to let go of causality, by introducing an error factor in throws. This was a fairly natural step since it felt inappropriate that, once a successful throw was discovered, the thrower would succeed every time (the world so far being completely determined).

Once error had been introduced, the student was no longer completely in control and aspects of randomness had to be addressed.

Macro level project

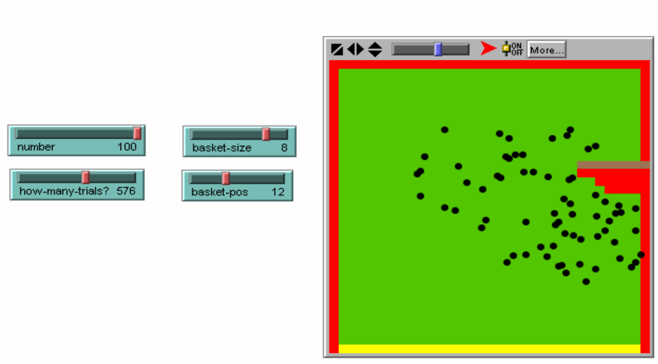

Figure 1: The right hand sliders again allow the student to change the size or position of the basket. The left hand sliders enable the student to decide how many balls should be thrown and how many repetitions are carried out.

Whereas at the micro-level the purpose of the task was to succeed in throwing the ball into the basket, at the macro-level the students were asked to design the court so a class of unknown children might find scoring neither too easy nor too difficult. The process of throwing was no longer under control of the student but was automated by the computer. This task sought to redirect attention towards the emergent distribution. We hoped that the student would have to let go of causality and consider the distribution of the throws. These ideas were incorporated into the macro level project (Figure 1).

At both micro and macro levels, the students can base their decision on various types of feedback, such as counters of goals, rate of goals, and three different types of graphs, namely average rate of goals per trial, a histogram of goals against position of basketball throw, and successful-shoots against trials (Figure 2)

We focus on the work carried out by two girls, Kate and Anna, (aged between 17 and 18 years), having already experienced throwing the ball into the basket at the micro-level. Overall, the girls worked for a period of four hours at the two levels.

We captured their on-screen activity on video-tape and transcribed those sections to generate plain accounts of the sessions. Subsequently we analysed the transcriptions in attempts to account for the students' actions and articulations. The excerpts in the next section are taken directly from transcriptions of the videotape.

AT THE MACRO LEVEL

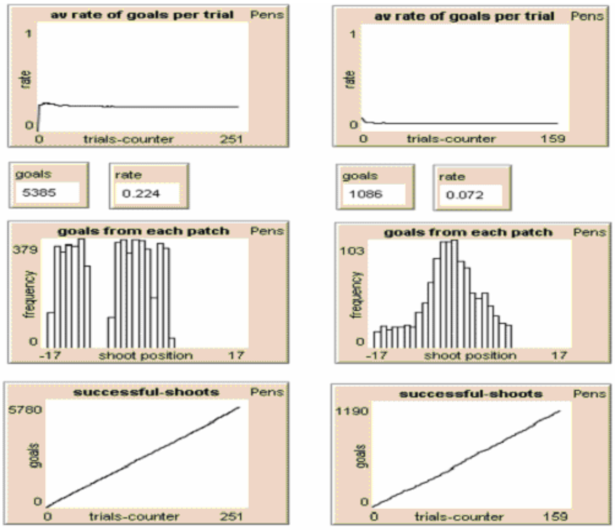

Figure 2: At both micro and macro levels the students have access to a graph of goals per trial, a histogram of goals against position of basketball throw each, and a graph of successful-shoots against trials

The two girls chose to work initially with just one screen-child, thus replicating the one thrower situation from the micro-level project. However, they had not anticipated not being able to control the position of that child in the way that they had controlled their own position in the micro-level project. Each time the child threw a ball, it threw from a different place, making it difficult for them to determine success rates in relation to pre-determined positions.

After they carried out 96 trials, Kate spontaneously recognised the random nature of the throwing positions:

1. Kate: That rate is not very constant… because it is in random places, I think.
2. Anna: We cannot judge because we have only one child… [long pause] …because he is only a child … and he doesn't know how to play it.
3. Kate: We have not got certain angle and speed.
4. Anna: We cannot control the throw of the ball.
5. Kate: The rate is very very low so… the line is very down. Almost zero… It is very constant... it is not a straight line.
6. Anna: Ehm…Can we exert any control? There are must be a position from where they can score best… Can we find it by standing only on a certain position?
7. Kate: …but we do not have a fixed shooting position.

The only control that Anna and Kate might have exerted was through the position or size of the basket. The basket however is the end of the process, the result of throwing. Perhaps then they did not construct the basket position as a control. Instead, they were confused by the lack of any control over the position of throws of the unknown child, which were perhaps more readily constructed as inputs or parameters.

Two hours and fifty five minutes into the activity with the microworld, the theme of lack of control had widened to encompass the situation as a whole and once more Kate associated the behaviour with randomness:

8. Res: Ok, so you cannot improve the scoring rate?

9. Anna: Not really.
10. Kate: You could but you can …randomly, you cannot plan to. It depends on the height they are really and how they throw the balls, and we do not know how they throw the balls.
11. Anna: It is like training different children every time and you do not know how they react because they can really… The only thing you can to is to make the basket size bigger.
12. Kate: We can make the basket bigger or move the basket, but the children do not know that we can do that.
13. Res: Can you improve the performance or control it in a way?
14. Kate: Not really, because there are so many different position… it is only a small number of people, but the same on different positions… so you don't know where they are going to stand… unless you fix their positions... .it's by chance.
15. Kate: It is a bit annoying really, because there are too many variables. Everything is… different every time...the people being standing in different places every time…and they have different heights every time… Everything happens by chance.

The girls struggled to attend to all these variables or agents, which function in parallel and interacted independently at a low level. They were unable at this stage to perceive any underlying distribution nor to discriminate and make strong connections between levels. Later though, when trying to interpret a histogram based on 80 trials (with 100 players), they called upon knowledge of bell-curves to begin to recognise features of the distribution.

16. Kate: That's different than before.
17. Anna: That's weird.
18. I ask her to explain what she means. Anna shows with the cursor on the histogram.
19. Anna: That's the best position. That graph… the positions towards the middle are better than the ones to the other sides.
20. Kate: Normal distribution.
21. Res: Normal?
22. Kate: Ehm… may be different because we didn't take too many trials.
23. Anna: And because there are quite… always change… so it cannot be the same every time anyway.

At one stage the researcher asked the girls about the various types of graphs, and we see them interpreting features of the graphs, resulting from 98 trials with 100 players.

24. Res: What about the three graphs?
25. Anna: They are constant. The graphs are constant, straight lines… (referring to the successful shots and average rate of goals per trial graphs).
26. The researcher asked about the goals for each patch graph.
27. Anna: That makes sense really… that makes sense because the further away from the basket the hardest it is to score, but afterwards… where it goes down

are those who scored and the rest would be under the basket or nearly under the basket?

After three hours and twenty eight minutes of activity, the girls, whilst interpreting a histogram based on 350 trials with 40 players, noticed that the distribution can sometimes be skewed.

28. Res: Ok, when do you think we have a skewed distribution?
29. Anna: When you have few trials and few people, and then the distribution is skewed…. or few trials really… actually.
30. They carry out 35 trials with 40 players/shooting positions.
31. Kate: The distribution is not skewed now, but before it was.
32. Res: You told me that with less trials we will have a skewed distribution, but we don't have. Why?
33. Kate: Maybe that one was by chance or… I do not know.
34. Res: What do you mean "by chance"?
35. They replicate the same simulation several times.
36. Anna: Anyway, it always changes…
37. Kate: Yes it is different every time and the rate is much higher than it was the last time… because the height is continually changing… so… I think it is just chance.

Kate and Anna appeared not only to be recognising macro-level features but they were relating those features to how many throws were being used. The notion that "the more trials, the less skewed the distribution" was articulated and seen as something that by chance might not happen. Indeed this pattern was not seen as something that could be controlled.

38. Res: What do you think affects the rate and the shape of the distribution?
39. Anna: I don't think it is controlled to be fair when you've got a bit number of trials and people… because the people being standing in different places every time… and they have different heights every time… so you can't control it.

Control seemed to seen in strict deterministic terms, even though they had perhaps begun to articulate earlier a sense of being in partial control of the distribution through the number of throws

KEY DESIGN PRINCIPLES AND THE NEW BASKETBALL MICROWORLD

The episode taken from work with the macro-level project gave us some insights that have allowed us, in the spirit of design research, to generate some new conjectures, which we are currently building into the next design of the emerging microworld (Figure 3). The features of this microworld related to the key design principles. This microworld will be used with students later this year.

The students continued to see control as embedded in the action of throwing (lines 2-4, 6-7, 10-11, 14, 37) but nevertheless articulated ideas that from our perspective sounded like a form of stochastic control (lines 19-22, 24-27, 29-32). They recognised the bell-curve feature (lines 19-20) and abstracted the notion that "the more trials, the less skewed the distribution". We would hope though that a more fine-grained appreciation of the nature of control at the macro-level might be constructed. We envisage that a more gradual letting go of deterministic control might allow the students to construct a relationship between the degree of control and the spread of the distribution.

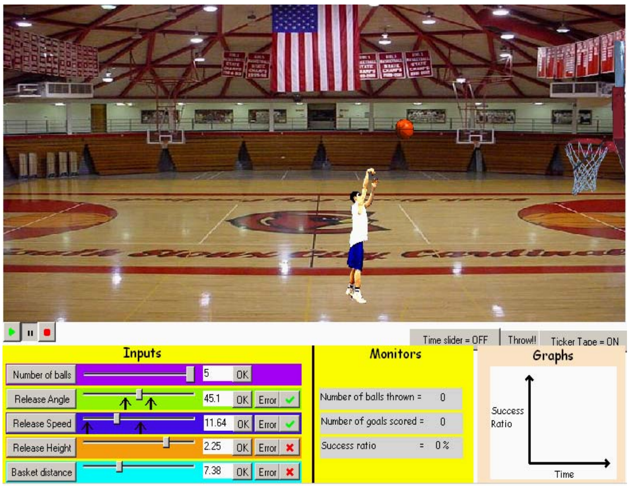

Figure 3: The sliders allow the student to select how many balls should be thrown, the release angle, release speed, release height and basket distance. The error button introduces size of error over the elements of randomness. Results are displayed in monitors such as number of balls thrown, number of goals scored and success ratio. Only the graph of success ratio against time by default is now presented in the Interface window. The other graphs of any variable that has error can be chosen through an options menu.

We also conjecture that the use of two separate models (micro and macro) creates something of an obstacle to shifting between the two levels and so our new design incorporates the two levels into a single project.

The macro-level project appeared to contain too much randomness, with too little control over how quickly that randomness was introduced (lines 15, 39). As shown above, the notion of control, or lack of control, was crucial. One aspect of the constructionist philosophy is to encourage the deepening of knowledge through the use of that knowledge (see for example Papert's *Power Principle* in Papert, 1996). In this instance, we wished to engage the students in using randomness in order to gain a better appreciation of the factors that were generating their feeling of lack of control. To put it succinctly, we wanted to give them control over the lack of control.

We therefore decided to embody error in a consistent way across the new single project. In this way, the error button would introduce an error on the elements of randomness, namely release angle, release speed, release height and basket distance. All variables would be fixed by default but with the option of adding error, which itself could be increased or decreased in size. When the error switch were set on, two marks would appear on the slider. These marks could be moved to represent the spread, and would be equivalent to two standard deviations from slider button. Thus, the students would have control over error size, which would be generated randomly from a normal distribution across the range of their sliders. A student might choose to have only one variable with error and then gradually increase that error before introducing a second variable with error. We believed that, with this additional control over control, students would be able to connect the deterministic to the stochastic in a more fine-grained way.

Throughout their use of the micro and macro projects, the discussion about the histograms of goals from each patch seemed to involve the students in beginning to conceptualise probability distribution as an emergent phenomenon. In contrast, discussion about the other graphs was at best trivial and at worst a distraction from our research agenda (lines 25-27). We have realised that some of the graphs are more important in terms of making connections between the levels. We have therefore decided to reduce the number of types of graphs to the most relevant ones but to offer those graphs for any or all variables. To help users focus effectively on critical ideas, we envisaged that students would be asked to select and look at a graph of any variable that might have error. In all of these cases students would have access to a range of graphs, such as a histogram of number of successes against position of scoring, histogram of frequency of successes against angle, speed or height. Our current plan is that the next iteration will take place during the Spring of this year.

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