Mejia-Ramos, J. P. and Tall, D. (2005), 'Personal and public aspects of formal proof: a theory and a single-case study', in D. Hewitt and A. Noyes (Eds), *Proceedings of the sixth British Congress of Mathematics Education* held at the University of Warwick, pp. 97-104. Available from www.bsrlm.org.uk.

PERSONAL AND PUBLIC ASPECTS OF FORMAL PROOF: A THEORY AND A SINGLE-CASE STUDY

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This study examines the construction of formal proof in undergraduate mathematics, within a framework of students' personal cognitive development and the view of proof in the mathematical community. The cognitive framework is based on a theory of three worlds of mathematics in which worlds of embodiment and symbolism in elementary mathematics are reconstructed as a world of formal definition and deduction. The view of proof in the mathematical community is framed by a distinction between logical formal proof, the logically-sound transformation of formal sentences, and mathematical formal proof, that is what mathematicians actually do to formulate and communicate their ideas. The theory is illustrated by a descriptive case study of an individual mathematics student, which is part of a wider multiple-case study.

COGNITIVE FRAMEWORK: THREE WORLDS OF MATHEMATICS

The theory used here arises from a study of the cognitive growth of individuals from child to adult (Tall, 2004; Tall & Mejia-Ramos, 2004). It contrasts the role of perception of objects in the world, actions on those objects, and our reflections on these perceptions and actions. Perceptions of objects become increasingly sophisticated, supported by an elaborated use of language to first describe, then define concepts, to subsequently use them in deductive proof (if this property holds, then that property holds) to give a mental world of imagination and thought experiment. This world of human perception and action is called the *embodied world*. Certain actions on objects, such as counting, are compressed into concepts, such as number, using symbols that act dually both as process (e.g. addition 3+2) and concept (sum, 3+2), a combined notion that is termed a *procept* (Gray & Tall, 1994). The developing world of symbols as procepts develops through arithmetic, algebra and symbolic calculus to give a *proceptual world* of calculation and symbolic manipulation.

Our interest occurs at a point where the properties that arise in these worlds of embodiment and proceptual symbolism are re-formulated in a new *formal world* of quantified set-theoretic definition and formal proof.

ASPECTUAL FRAMEWORK: FORMAL PROOF

The view of proof in the mathematical community has different aspects that give rise to different notions of proof in the formal world. While a *focus* on its logical aspect leads to the ideal *logical formal proof*, illustrated for instance in Whitehead and Russell's *Principia Mathematica*, a *focus* on its psychological and social aspects leads to *mathematical formal proof*, which is how mathematicians actually formulate their ideas and convince other qualified experts. While logical formal proofs are exhaustive, mathematical formal proofs are *contextual* in the sense that they assume an enormous amount of contextual knowledge (Stewart & Tall, 1977). At its best, formal proof reaches an equilibrium in these three aspects, but this is not always the case. This distinction resembles Hersh's (1997) discrimination between theoretical and practical mathematical proof, and Devlin's (2003) right-wing and left-wing definitions of proof. This distinction is also related to Raman's (2002) study of public and private aspects of proof, which refer respectively to public arguments (procedural ideas that provide what we call *external conviction*, i.e. a sense of conviction based on the belief system of a particular mathematical community), and private arguments (heuristic ideas that provide personal understanding and what we call *internal conviction*, i.e. a sense of conviction based on intuition and internal belief systems). Essentially, logical and social facets of proof correspond to its public aspect, while its psychological facet corresponds to what Raman (ibid.) has termed the private aspect of proof.

We are interested on the interplay of these aspects in the notion of proof developed by undergraduate students as they are introduced to the formal world of mathematics.

CASE STUDY

The present case study is part of a descriptive multiple-case study that has been carried out with a wider range of students. Its aim is to confront students' views of proof with the theoretical frameworks presented above. This study replicates Raman's (2002) method of data collection which consists of a task-based interview in which the participants are asked to prove a mathematical statement (*"prove that the derivative of an even function is odd"*), and to evaluate five different ready-made responses to that task. Raman's method of gathering data from participants' production and evaluation of proofs was adapted from a previous study conducted by Healy and Hoyles (2000) which focused on secondary students' conceptions of proof. This method has also been used by Selden and Selden (2003) in a recent study of undergraduates' validations of proofs. In each of these studies, the participants were asked to work on the production of a proof of a statement and evaluate, according to different criteria, a series of ready-made normative and non-normative proofs of that statement. This general method allows us to concentrate on participants' views of the meaning of proof, and the relationship between those views and their actual approaches to proof. While Raman's (2002) study was set to compare the views of proof held by entering university level students, with those held by their teachers, the present case study focuses on the interplay of the different aspects of a person's particular view of proof. Therefore, this case study offers a more in-depth analysis of the data collected through the task-based interview.

GRAD

Grad, our participant, is a Chinese male who recently completed a mathematics undergraduate program at an English university. Below we report the way in which he approached the task, and all of his commentaries about this approach. We also report on his views on each of the ready-made responses, as expressed throughout the interview.

Task:

Determine whether the next statement is true or false (explain your answer by proving or disproving the statement). *The derivative of a differentiable even function is odd.*

As he read the statement out loud, Grad drew a parabola in the air with his right index finger and conjectured that an even function cannot have an even derivative since its graph is symmetrical with respect to the *y*-axis (this time using his hand to denote a vertical axis). This argument was supported by an example of a quadratic function, represented by another finger-drawn parabola, the derivative of which was thought to be odd because it is always decreasing (a confusion probably generated by the downward movement of his hand as he started drawing the parabola). Without considering other cases, he concluded saying: "generally I think it's true, but [laughing] I'm not so sure".

After revisiting this graphical argument in an actual drawing, and correcting his earlier mistake, he was asked if he could prove that the statement was true. At this point, he wrote the definition of an even function, as well as the limit definition of its derivative. After a couple of minutes of silent thought, he said that he was trying to show that the limit denoting $f'(x)$ was "minus" the corresponding limit for $f'(-x)$. However, he declared that he did not know how to "open the brackets" inside these limits, which would allow him to use the definition of even function to "cancel out some part" of the limits, and finally show that they were inverse. Eventually he found the way to follow his plan through. However, since he was not completely sure about the validity of the steps in this algebraic procedure, he claimed to be only 50% convinced of the statement. As I suggested his symbolic manipulation was correct, he raised this percentage to 80%.

Response 1:

Consider the following functions and their derivatives:

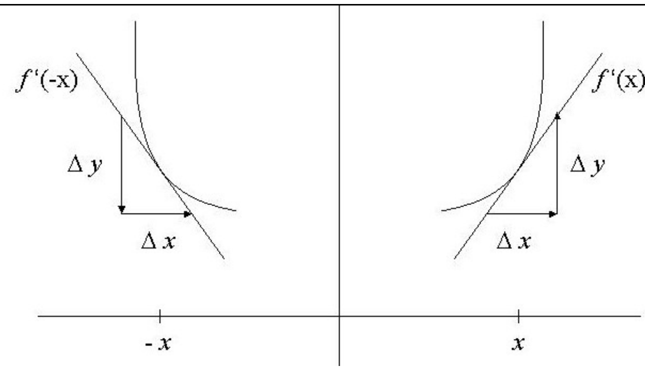
$f(x)=x$	odd	$f(x)=x^3$	odd	$f(x)=x^5$	odd
$f'(x)=1$	even	$f'(x)=3x^2$	even	$f'(x)=5x^4$	even
$f(x)=x^2$	even	$f(x)=x^4$	even	$f(x)=x^6$	even
$f'(x)=2x$	odd	$f'(x)=4x^3$	odd	$f'(x)=6x^5$	odd

Note that for all even functions, the derivative is odd. We could continue for all powers (n = 7, 8, 9, ...), thus the claim is proved.

Shortly after reading this response, and throughout the interview, Grad declared that it was not a convincing proof, as it was "just try and go" and the argument only applied to a very limited subset of differentiable even functions. With regards to the mark it would receive, he said it depended on the educational setting: while at undergraduate level this response would get 0/100, at high-school level it would get 80/100. For Grad, this difference did not rely on the level of rigour or persuading power of the argument, but on the knowledge displayed by it. In other words, he suggested that knowing how to differentiate functions, and having a limited notion of the concept of even function, was satisfactory for a high-school student, but not for an undergraduate student.

Response 2:

If $f(x)$ is an even function it is symmetric over the *y*-axis. So the slope at any point *x* is the opposite of the slope at –*x*. In other words $f'(-x) = -f'(x)$, which means the derivative of the function is odd.



Grad initially said that Response 2 "looked fine", while an immediate second reading led him to question the level of rigour of the argument and conclude that, although he could sense it was true, the response was only 50% convincing. When these apparently contradictory opinions were probed, he said that every claim in the argument was true, but they needed to be further explained. Accordingly, Grad thought that this response would get less that 50/100 at the university level.

However, he also thought that at the high-school level this was a "totally acceptable and correct answer". Furthermore, later in the interview he claimed that Responses 2 and 4 were "very clever", and selected them as the arguments he preferred. At this point of the interview he also chose Response 2 as the one that portrayed the best understanding. When asked to explain the incompatibility between the high level of understanding portrayed by this response and the low mark given at a university, he replied:

Grad: This statement [pointing at the second statement of Response 2] is visually true, that's not… it needs to be proved… Yeah… But he can show that he can judge this statement [pointing to the task] very convincingly, *for himself*, but not very convincingly for others.

At the end of the interview, he concluded:

Grad: If this guy can prove this statement [pointing back at the second statement of Response 2], that would be the best proof."

Response 3:

Want to show if $f(x) = f(-x)$ then $-f'(x) = f'(-x)$.

$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h}$ by the definition of the derivative.

$f'(-x) = \lim_{h\to 0} \frac{f(x-h) - f(x)}{h}$ since *f* is even.

Let $t = -h$

$f'(-x) = \lim_{t\to 0} \frac{f(x+t) - f(x)}{-t}$

$f'(-x) = -\lim_{t\to 0} \frac{f(x+t) - f(x)}{t}$

$-f'(x) = f'(-x)$, as required.

Grad thought that this was a "very ordinary" response, one people would normally give: straight from the definition. Although his proof was virtually equivalent to the one in Response 3, he thought that the latter was "more formal" and "very convincing". However, he also stated that this response is not totally satisfactory at university level:

Grad: Because you need more words… verbal expression about how you analyse a question, how you applied the definition, how you solved it… err… [reading from Response 3] "want to show if this then this"… well, there is an explanation missing. Why is this equivalent to the statement? Something like that. Well, maybe it can depend on the marker [laughs].

Response 4:

Given $f(x)$ is even, so $f(x) = f(-x)$. Take the derivative of both sides.

$f'(x) = -f'(-x)$ by the chain rule. So $f'(x)$ is odd.

Grad immediately recognized and praised the economy of this response. He said it was a "more standard solution", the kind that would be given and appreciated by a teacher. As a result, he thought that a teacher would give it full marks. Later in the interview he claimed that Responses 2 and 4 were "very clever", and selected them as the best arguments. However, he appreciated Response 4 in a very different way:

I: Which of these arguments [Responses 3 or 4] do you think is more convincing?

Grad: Well… this [pointing at Response 4]

I: Why?

Grad: Less steps, the less mistakes you can make…err… less assumptions made… yeah, straight-forward.

Response 5:

f is even, so $f(x) = f(-x)$. Multiply both sides by –1 $-f(x) = -f(-x)$ Factor in –1 $f(-x) = -f(-x)$ *f* is even, so we can substitute $f(x) = f(-x)$ $f(-x) = -f(x)$ Take the derivative of both sides $f'(-x) = -f'(x)$ So f' is odd, as required.

Although he doubted for a couple of minutes (and at one point thought it might be true), Grad said that the argument was confusing and not convincing at all. He eventually concluded that it was completely wrong, and just an attempt "to fool the teacher".

DISCUSSION

In the above description, we have intentionally omitted reference to the cognitive and aspectual frameworks so that the reader can form his or her own views. We can now reveal our analysis of these responses in terms of the theoretical framework.

Grad's initial response to the **task** is a perfect example of an argument in the embodied world of mathematics. In this episode, Grad justifies his claims by carrying out a thought experiment supported by an enactive representation of a generic example. This heuristic idea gives him a sense that the statement is true, but does not give him a strong sense of conviction. Contrastingly, Grad's second approach, a response to the requirement of proof, was essentially symbolic and procedural: he initially symbolized what was given and what was wanted, and then he looked for a valid algebraic procedure to go from the former to the later. The strong influence of external approval on his level of conviction indicate that this argument provided Grad with an external sense of conviction. Grad's solution illustrates the way in which his experiences in the embodied and proceptual world (which develop much earlier than the formal world) strongly influence his approach to formal proof.

Response 1 illustrates the general cognitive strategy of evaluating a statement in a series of particular examples in the proceptual world of symbolism. It is a private argument that does not explain why a statement holds true, but can provide some level of internal conviction. In mathematics, this strategy is developed in the proceptual world through thought experiments on generic examples. However, this strategy is no longer appropriate in the formal world, where objects belong to defined categories. This characteristic is strongly stressed by teachers, but the development of the formal world is not smooth: students find it hard to abandon the general cognitive strategy and many learn to disregard it as a means of formal proof only to please authority. Correspondingly, Grad, who had initially relied on a thought experiment, judged Response 1 in terms of its public aspect (i.e. its level of rigour and its power to persuade a sceptical mathematician), taking no notice of its ability to internally convince a student, or its role in the eventual development of a more formal argument.

Response 2 is a good example of an embodied proof as its main argument rests on a thought experiment with an embodied, generic example. Overall, Grad expressed two contrasting opinions in relation to Response 2. On the one hand, he regarded it as a very clever, true argument that portrayed understanding and would get the highest marks in high school. On the other, he thought it lacked rigour, needed further explanation, and would get less than 50/100 at a university. Furthermore, Graduate A believed that, while failing to provide external conviction, Response 2 was internally convincing. Most importantly, he intimated that the argument did not qualify as a proof, even though it was considered to be true, insightful, and internally convincing. This episode suggests that, even though he appreciates the value of private arguments, Grad's notion of proof focuses on its public aspect, and relegates its private aspect to a less fundamental plane.

Response 3 is praised by Grad; its public aspect provided him with a strong sense of external conviction, but he also denounced its shortcomings as it fails to explain every argument to a strict marker. Later in the interview, comparing Responses 3 and 4, he acknowledged that any proof could be further refined by adding more explanations. Taken with his opinions on Response 2, this shows that Grad is aware of the social aspect of mathematical proof and its dependence on the context in which it is presented.

From his comments on **Response 4**, it becomes clear that Grad judged the quality and convincing power of the response by taking into account only its public aspect. He regarded Response 4 essentially as a concise, correct procedure, without taking into consideration the mathematical meaning of the argument.

Despite Grad's previous focus on the public aspect of proof, he approaches the formal-looking argument in **Response 5** by carefully checking each one of its claims. This may suggest that through his educational experience he has developed a view of proof according to which the value of a proof ultimately relies on the validity of the step-by-step deductions (i.e. procedural ideas). Therefore, while he might think that the ritualistic format is to some extent necessary (as expressed in his evaluation of Response 2), he also considers it insufficient on its own.

In this analysis, we find Grad personally coming to terms with formal ideas by using embodied thought experiments, yet aware of the public aspect of formal proof and the need to build a proof in a way that will be approved by the mathematical community.

The full study (Mejia-Ramos, 2004), places Grad's response in context with six other graduate mathematics students who take part in the study. Each of these students has his or her own view of proof in mathematics. This may be similar to Grad's way of building from embodiment to give personal meaning, and learning to be part of a public community where communication is accomplished through a mathematical form of proof that combines logic with appropriate references to already established formal theorems. Some students build more on the calculating power of symbols in arithmetic and algebra from the proceptual world rather than from the thought experiments of the embodied world. Some move part way to the mathematical community, accepting the public need to share in the formal presentation of arguments without being able to reconcile personal and public aspects of mathematical thinking, while others espouse not only the *form* of mathematical formal proof but the meaning of being a fully fledged member of the mathematical community.

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