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ARITHMETIC EQUATIONS: CONVENTIONS AND ORDERING

Dave Hewitt

School of Education, University of Birmingham

This study looks at two Year 7 classes in contrasting schools and at ways in which students interpret and write arithmetic equations in formal notation and their reading of equivalent written word statements. Students appear, not surprisingly, to be able to read formal notation before they have confidence with writing it and tend to apply conventions in a context dependent way. Many students attempted to write equations down in the order in which operations were to be carried out. This led to symbolism which did not conform to standard conventions and I make a suggestion of how teachers might help students write and read equations in formal notation whilst explicitly addressing the issue of order of operations.

INTRODUCTION

Several studies were carried out in the 1980s identifying students’ difficulties with algebra (Booth, 1984; Küchemann, 1981; Dickson, 1989). Some of these studies and others have suggested that teaching approaches have contributed to some of the difficulties students experience (MacGregor and Stacey, 1997). Coles and Brown (2001) and then Ainley et al. (2003) have looked at approaching algebra in a way where the algebra serves a purpose and students learn algebra in a meaningful way, through enhancing an activity rather than being an end point in itself. Such studies have indicated that many students are capable of algebraic activity when they experience pay-off to working with algebra. Carraher, et al. (2001) have looked at introducing activities to younger students to see whether eight to nine year olds are capable of solving algebraic tasks.

Alongside these studies has been a debate about whether certain activities should be classified as algebra or arithmetic. For example, the equation $2x+7=15$ can be solved by trying out different numbers for x until twice it plus seven equals 15. It can be argued that such activity is still essentially arithmetic. Filloy and Rojano (1989) have talked of a didactic cut between arithmetic and algebra when the unknown appears on both sides of an equation. This makes it less likely that students would test a series of numbers as in the case above. Herscovics and Linchevski (1994) challenged this definition by saying that a divide between algebra and arithmetic is not so much to be established through looking at attributes of an equation but with human activity. They suggest it is when a student has to operate spontaneously on the unknown that arithmetic activity changes into algebraic activity.

Mason (1993) also looks at mathematical activity and suggests that algebra comes through seeing the general in the particular. It is the activity of working with generality which is quintessentially algebraic and this does not necessarily involve working with algebraic symbols such as *x*. Elsewhere (Hewitt, 1998; Hewitt, in press) I argue that in order to be able to carry out some number tasks, such as being able to say number names (e.g. 1345, which I suggest is not a matter of reciting from memory) or carrying out calculations (e.g. 15×6, which is not just a matter of recall) someone already needs to use an algebraic structure in order to carry out those tasks. Thus algebraic activity necessarily precedes arithmetic activity. As such, the issue of working algebraically (which I claim all students can do) is distinct from working with a precise given formal system of notation (with which many students have difficulty).

There have been many studies involving algebraic activity within computer notation systems such as *Logo* [1] (for example, Clark and Redden, 2000) or spreadsheet environments (for example, Sutherland, 1993). Although I consider all of these involve algebraic activity within formal systems of notation, they are different systems of notation to the one which is traditionally used for pencil and paper school algebra. There may well be arguments about whether this traditional system may change in the light of technology but, whatever may happen in the future, it is clear that the traditional algebraic notation system is alive and well and very much part of the present school curriculum. A significant difference between a notational system which is used within computer systems and traditional algebra, is that computer environments require students to enter expressions in a linear form (with the notable exception of *Working with Equations* [2]). Expressions are entered via the keyboard and as such have an inherent temporal ordering to them. Software such as *Derive* [3] will translate keyboard entered expressions to traditional algebraic notation, however the student still enters the expression in a different notational system to the traditional one.

In this study I have concentrated on traditional algebraic notation and in order to separate this from issues relating to the use of letters I used equations involving only numbers and wanted to see how students: (a) interpreted the formal notation system when deciding whether a given equation is correct or not; (b) wrote equations within that system; and (c) how their interpretation of equations in symbols compared with equivalent written word statements.

TWO SCHOOLS

The two schools I will report on within this paper are contrasting in that School A is a selective boys grammar school (with 98% of students gaining five or more grades A*-C at GCSE) and School B is a mixed comprehensive school (42% of students gaining five or more grades A*-C at GCSE). One ‘low ability’ year 7 class from each school (relative to other classes within the school) was involved in the study. There were 28 students in the School A class and 26 in the School B class. These schools were part of a larger study involving seven schools.

DESCRIPTION OF THE STUDY

The classes were given two tasks. The first task involved a set of 17 arithmetic equations using conventional algebraic notation. Such conventions as using a horizontal line for division and not using a multiplication sign before a bracket were used. Some of these equations were arithmetically correct, such as $2(3+2)=10$, and some were incorrect, such as $\frac{8}{3+1}=4$. Within the analysis of results I assumed the conventions of order of operations held. For example, $1+3\times 2=8$ was considered to be an incorrect equation. Some equations involved a single number on the right-hand-side (RHS) of the equation (as in the examples above), whilst others involved an expression on the RHS (such as $2\times 3=\frac{12}{2}$). Likewise for the LHS. The number of operations involved in the equations ranged from one (four equations) to three (one equation). The task for the students was to put a tick next to each equation they thought was correct, and a cross next to those equations they thought were wrong. This task I will call the *symbol* task.

The second task involved 17 word statements. This task I will call the *word* task. These 17 statements were designed to be equivalent to their symbolic counterparts. Thus equation 1 in the symbol task was $3+4=7$ and statement 1 in the word task was *two add three equals five*. Both of these have the same underlying mathematical structure, they have one addition on the LHS of '='/*equals* and have a single number on the RHS. Both are arithmetically correct. The word task was two-fold for the students: first they had to state whether they felt the word statement was correct or whether it was wrong; second they were asked to write down each statement, irrespective of whether they thought it was right or wrong, as a mathematical equation. The symbol task was carried out by the students before the word task. I will report here on similarities and differences between the two schools in relation to the division and multiplication signs, and phrases used within the word task. This will lead to brief discussions concerning students' interpretation and use of conventions; fractions; and how to help students' understanding of order when reading and writing equations in formal notation.

DIVISION AND MULTIPLICATION

School A had a very high success rate in deciding whether the symbolic equations and word statements were correct. They gained 93% success with the symbol tasks and 98% success with the word tasks. There were no particular equations/statements where there was a significant difference in success between the symbol and word versions. The students appeared to be able to interpret formal algebraic notation yet rarely used certain aspects of that formal notation themselves. For example, in those word statements which involved a division, a '÷' sign was used 98% of the time rather than a horizontal line when they expressed a word statement as an equation.

Thus, there appeared to be confidence in reading such notation before they had confidence in writing such notation. The convention that a multiplication sign is not used immediately before a bracket seemed to be interpreted by all but two students whilst almost no-one used that convention when writing their own equations to represent the word statements (a multiplication sign was written on 86% of possible occasions immediately before a bracket). Since the symbol task was completed first, students had already seen and interpreted the horizontal division sign and the non-appearance of the multiplication sign. So, a convention appeared to be successfully interpreted when reading equations before the students had confidence in using that convention themselves.

Looking at the particular equation $1+3\times 2=8$, the vast majority of students from the School A class did interpret this by doing multiplication before addition and so their interpretation had a non-left-to-right ordering. The equivalent word statement, *two plus one times three equals nine*, was deliberately written as to not indicate within the wording a preferred order. The vast majority of students from the same class (79%) this time read the statement in a left-to-right order and so did not transfer their implicitly understood convention of multiplication before division in the symbolic equation to the context of the word statement. A similar phenomenon happened with the fourth equation above. So it seems as if the use of the implicit convention summarised as BODMAS or BIDMAS in many schools, is context dependant on the notational system being used. In the School B class, left-to-right reading was consistent within both symbol and word tasks and so the issue of context dependency of this convention did not arise.

School B found the symbol task to be harder than the word task (63% success rate compared with 85%). There were particular difficulties with the formal division sign of a horizontal line. Apart from the three equations in the symbol task mentioned in the previous section on left-to-right reading, the next seven equations incorrectly answered by students were all the ones which involved the horizontal division sign. The '÷' sign was also used in nearly every equation written by students themselves. One student wrote on her symbol sheet "I do <u>not</u> understand the Fraction Sums" and did not answer any question which involved a division. The use of the word *fraction* gives me a sense that the horizontal line is associated with fractions and not with division, perhaps seen within a single entity (a fraction) rather than an operation on two numbers (division).

It appears that the horizontal division sign is not naturally used by these students no matter what their ability level. Of course, the horizontal line itself will be very familiar to all the students within the way fractions are written. It is not that this is a novel notational symbol for them. However, fractions can sometimes be considered as a whole rather than attention given to its parts. A fraction is an expression of a division as well as the answer to that division. If it is rarely viewed as the former then it will not be seen as the latter either. Instead, I suggest that by viewing a fraction solely as a number the horizontal line is never abstracted from the whole as having a meaning on its own.

PHRASES USED WITHIN THE WORD TASK

Words in English are written left-to-right and so there is a natural temporal order created as someone reads. However, some mathematical expressions are difficult to express within such a linear order, hence non-linear symbolic conventions have been created which make use of vertical placement and well as horizontal placement. When I wrote the word tasks, there was no ambiguity if only one operation was involved. However, when an expression contained more than one operation, I made choices about how to express order within the strict left-to-right system of the English language. Below are three examples of the choices I made:

Five add one and then times two equals twelve

Six divide by the result of one add two equals three

Two plus one times three equals nine

The first involved the word *then* to indicate an implicit bracket in the calculation stated *prior* to that word. The second involved the phrase *the result of* to indicate an implied bracket in the calculation *following* that phrase. The third involved no additional words or phrases and so was deliberated left ambiguous.

With the School A class the terms *then* and *the result of* seemed to clarify the order and this resulted in a high success rate in judging whether a statement was right or wrong. It also led to appropriate symbolic equations being written for the word statements. Those word statements which were ambiguous were interpreted generally with a left-to-right order of operations. With the School B class however, several other phenomena appeared in the students' attempts to write symbolic equations equivalent to the word statements. For these students, the phrase *the result of* appeared to trigger the need for a calculation to take place. There were two word statements including this phrase, and for attempts at writing equivalent symbolic equations for these nearly half the attempts involved students carrying out a calculation. For example, $6 \div 3 = 3$ was written for *Six divide by the result of one add two equals three*. For some students the phrase not only meant that what followed *the result of* was to be carried out first, it was also to be written first. So, one student wrote $1 + 2 = 3 \div 6 = 2$ for the same word statement. *One add two* followed the phrase *the result of* and so this was written first. However, this led down a path where it was difficult for this student to continue expressing the whole equation with conventional notation. My interpretation is that she continued to try to express *six divide…* but had already got $1 + 2 = 3$ written down and writing $6 \div$ after this would not fit in with the convention of *number-operation-number* and so wrote the division sign first to fit in with this convention. Thus, having started how she did (whether she actually carried out the operation as she did or just wrote $1 + 2$ is not so significant), she found herself in a situation where she could not continue writing the rest of the equation in a conventionally correct manner. By the time she had written $1+2=3\div 6=$ it could be that her attention was with what she had written rather than the original word statement and so she wrote the 'correct' answer of 2. A similar situation occurred with a boy who wrote $6+1-8=3$ for *three equals eight subtract the result of six plus one*. Again what followed the phrase *the result of* was written first and an equivalent situation then followed, this time with subtraction rather than division. So errors in the order of how division and subtraction were written (resulting in *goes into* and *subtract from*) may not have been due to a conceptual misunderstanding of the operation and how it was written. Instead the understandable decision to write down first the operation which was to be performed first, and the convention that they would continue writing to the right of what was already on the paper, led to an inevitable deviation from conventional notation.

So there can be a dilemma between writing in a strict left-to-right order and wanting to write down operations in the order in which they are to be carried out. One idea to help with this dilemma is for a teacher to encourage the writing of expressions in order whilst still maintaining the formal conventions of the notation system. For example, the equation $4\left(\frac{2(x+5)-3}{7+1}+6\right)=13$ might be written an operation at a time in the following order:

1. $x+5$
2. $2(x+5)$
3. $2(x+5)-3$
4. $\frac{2(x+5)-3}{}$
5. $\frac{2(x+5)-3}{7+1}$
6. $\frac{2(x+5)-3}{7+1}+6$
7. $4\left(\frac{2(x+5)-3}{7+1}+6\right)$
8. $4\left(\frac{2(x+5)-3}{7+1}+6\right)=13$

Note that one operation is written at a time, in order of the operations, except for stages 4 and 5. With stage 4, only the operation is written with a temporal break afterwards. This is then followed by stage 5 where the expression $7+1$ is written quickly as a whole, to indicate that this can be viewed as a single object. This can help students begin to see the order of operations as they see such equations evolve in the order in which operations are carried out.

SUMMARY

Students' learning of formal notation appears to mirror that of learning the written text of their first language: confidence with reading the notation seems to be in advance of confidence with writing that notation.

Certain notational conventions such as multiplication before addition (unless brackets say otherwise, of course) were applied within symbolic equations but not within written word statements. Familiarity of the horizontal line for division was also context dependent, it was known to students within the context of a fraction (as part of a whole) but many students did not appear to know how to interpret it within the context of a symbolic equation (as an entity in its own right) and it was not used naturally by students within this study, no matter what their ability.

Several students naturally wanted to write down first the operation which was to be carried out first, yet sometimes the first operation does not appear first in a strict left-to-right reading of formal notation. I have suggested a way in which teachers can help with this issue by writing expressions in the order of operations even though sometimes this means writing in a non-left-to-right order. In this way the order of operations can be observed by students in time, even though they do not appear in order in space. This temporal ordering can assist students in beginning to read order within the spatial form of formal notation.

NOTES

1. A version of *Logo* called MSWLogo can be obtained from www.softronix.com/logo.html

2. *Working with Equations* is one program within *Mathematics Multimedia School* from Plato Learning (UK) Ltd, Statesman House, Stafferton Way, Maidenhead, Berkshire SL6 1AD. See www.platolearning.co.uk/

3. *Derive* is produced by Texas Instruments and is available from Chartwell-Yorke, 114 High Street, Belmont, Bolton, Lancashire, BL7 8AL. See www.chartwellyorke.com/

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