Asghari, A. H. (2005), 'Examples: generating versus checking', in D. Hewitt and A. Noyes (Eds), *Proceedings of the sixth British Congress of Mathematics Education* held at the University of Warwick, pp. 25-32. Available from www.bsrlm.org.uk.

EXAMPLES: GENERATING VERSUS CHECKING

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The purpose of this paper is to draw attention to the problems involved in 'generating' an example of a defined concept. We show that there are certain interactions between generating an example and checking the status of something for being an example.

INTRODUCTION

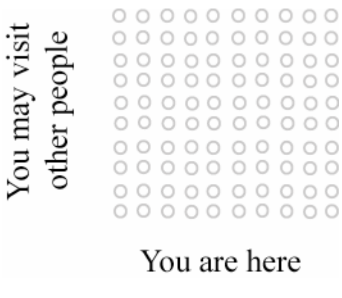
Using examples is so blended with our standard practice of teaching mathematics that what is written about the importance of using examples seems to be an expression of triviality. However, to the very same extent that making use of examples seems trivial and mundane, choosing a suitable collection of examples is problematic. It seems that any choice of examples bears an inherent *asymmetric* aspect, i.e. while for the teacher they are examples *of* certain relevant generalisations transferable to other examples to be met, for the students they could remain irrelevant to the target generalization. When the intended generalization is a concept, usually accompanied by a definition, this divorce of examples from what they exemplify is mainly shown in the literature by examining how students tackle *checking* problems, i.e. checking the status of something for being an example.

Contrary to the widespread standard teaching practice in which new concepts are introduced by and through teacher-prepared examples accompanied by his or her commentaries on what is worth considering in the prepared examples, there are a few and still experimental non-standard settings in which students are encouraged from the outset to *generate* their own examples. A case in point is Dahlberg and Housman's (1997) study. Dahlberg and Housman introduced students to a new concept "in an environment requiring self-generation and self-validation of instances of the concept". In detail, in a one-to-one interview situation, they presented eleven third and fourth year undergraduate mathematics students with the definition of a concept that they had not been taught in previous courses. Then they asked the students to generate their own examples. Dahlberg and Housman inferred that the students in their study who generated their own examples were more effective in "verification" (checking) and determining the validity of stated statements about the concept involved. However, confining themselves to making a distinction between those who generated their own examples and those who did not, Dahlberg and Housman paid no attention to the nature of generating an example per se. The latter issue is what we concern ourselves with.

BACKGROUND

This paper is based on a wider study aiming at investigating students' understanding of equivalence relations (Asghari, 2004, a, b). The Participants were quite varied in terms of age and educational level: the youngest participant was about fourteen years old (a middle school student) while the oldest one was about twenty- eight (a postgraduate student); the participants comprised four middle school students, four high school students, one first year politics student, one first year law student, six first year mathematics students, two second year physics students, two second year mathematics students, one second year computer science student and one postgraduate student in mathematics. Nearly all had no formal previous experience of equivalence relations and related concepts usually used to define it. In a one-to-one interview situation, each student was introduced to the definition of a 'visiting law', which was originally designed while having the standard definition of equivalence relations in mind (see below). Then each student was asked to give an example of a visiting law on the prepared grids (see below).

> A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.
>
> A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.
>
> A visiting law must obey two conditions to satisfy the mad dictator:
>
> 1. When you are in a particular city, you are allowed to visit other people in that city.
>
> 2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.
>
> The dictator asks different officials to come up with valid visiting laws, which obey both of these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below.



The timing of the interviews and questions were contingent on students' responses. In particular, depending on students' responses, some of them were asked to check the status of certain pre-prepared figures for being an example, and some were not.

The situation

In this section we informally outline the situation in terms of equivalence relations and related concepts. To do so, let us use the eloquent, but still informal, account of equivalence relations given by Skemp (1971). To elaborate the idea, he starts with methods of sorting the elements of a parent set into sub-classes in which every object in the parent set belongs to one, and only one, subset (a *partition* of the parent set). He (ibid, p.174) considers two sorting methods: first, starting "with some characteristic properties, and form our sub-sets according to this"; and second, starting "with a particular matching procedure, and sort our set by putting all objects which match in this way into the same sub-set". The particularity of this matching procedure is in its "exactness", i.e. having an exact measure for the sameness; a necessity that if it is achieved, the matching procedure is called an *equivalence relation*. The exactness of the matching procedure also accounts for the *transitive* property. And the importance of the latter is that "any two elements of the same sub-set in a partition are connected by the equivalence relation" (ibid, p.175).

The matching procedure- as Skemp uses it- could shed light on our task, where two cities are matched together if all their visiting-cities are the same, or two columns are matched together if they have the same status in each row.

Having been acquainted with the task let us turn to certain theoretical elements before articulating the involved *relation* between generating and checking.

DEFINITION FOR CHECKING vs. DEFINITION FOR GENERATING

Quoting Molland, Pimm (1993, p.266) pointed out a distinction between definitions of curves by *genesis* and definitions by *property*. According to Pimm, "definitions by genesis involve telling you what you have to do to produce the curve, whereas definitions by property involve specifying a property that the curve has." Moving away from this historical clear-cut distinction of two kinds of definitions of curves, and glancing at any randomly chosen mathematics textbook, it appears that mathematical definitions have turned to the latter, that is to say, they are definitions by property. However, this turn does not negate the possibility of making use of a definition to *generate* examples of what it has defined. For example, let us see the following definition of an even integer:

An even integer is an integer of the form 2n for integer n.

As it can be seen this definition is a definition by property, but it can be used to generate a specific even integer by replacing n with any specific integer. On the other hand, the given definition serves one of its prime purposes too, that, it can be used to *check* whether a given specific integer is even or not.

It is worth emphasizing that contrary to the distinction between definition by genesis and definitions by property, this distinction, definition for checking and definition for generating is not intended as a distinction between definitions per se; it is not a distinction of kind. It merely points to possible ways of making use of the same statement.

Let us invite you to a thought experiment (it is worth saying that the thought experiments used in this paper are firmly rooted in the analysis of students' experiences). Suppose you mention a certain definition to a student who is not aware of the possibility of making use of the given definition for generating an example of what the given statement has defined. Moreover, you ask her or him to *give* an example of the given definition. Thus, your student is more likely to choose something from the possible *universe of discourse* (if he or she is aware of that universe) and use the given definition for checking the status of the chosen object as an example. As an example, let us use our even integer again. To give a specific example of an even integer, the just-mentioned student is more likely to choose an integer (if he or she is aware that an even integer first and foremost is an integer) and then check whether it is even or not.

This situation occurs in the present study where students encountered a definition (definition of a visiting-law) and they were asked to give an example of the given definition. However, before presenting some data, where this functional distinction emerged form, we still need to clarify what generating an example mean.

GENERATING AN EXAMPLE

Let us give a concrete example. Consider the following question:

Give an example of a prime number.

For those that are familiar with prime numbers, *giving* an example is only a matter of *choice*, say, 3.

Now consider the following question:

Give an example of a 3-digit prime number.

This one is not one of those that student usually have at their disposal. Maybe the simplest way of giving such an example would be choosing a 3-digit number and then *test* it to see whether it is a prime number or not.

Eventually, we have the wonderful way of *generating* prime numbers by using the sieve of Eratosthenes (The following explanation of the sieve is based on Conway and Guy, 1996, pp. 127-130). Here what we act upon does not belong to the same *universal set* (say natural numbers) as the required example anymore; it is a linear or usually a tabular array of numbers. In its linear format we 'write down the numbers in order, putting 1 in a box to show that it's the unit'

$\boxed{1}$ 2 3 4 5 6 7 8 9 10 11 12 13 ....

As it can be seen our first *figural act* has been determined by our decision to put 1 in its special class. Our next figural act is 'circling the first remaining number, which is 2, and striking out every second number thereafter':

$\boxed{1}$ 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9 ~~10~~ 11 ~~12~~ 13 ....

'Circle the next remaining number, namely 3, and strike out all subsequent multiples of that number'

$\boxed{1}$ 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~ 13 ....

‘If we continue in this way at each stage, circling the first remaining number and striking out its higher multiples, the numbers we circle will be the prime numbers.’

As it can be seen all these figural changes are *regulated* by the definition of prime numbers as numbers that are bigger than 1, but not the product of smaller numbers. Furthermore, all of these figural changes can be done without any attention to the meaning of prime numbers, composite numbers or whatever else that this sieve has been based on it. That is what makes those changes figural.

Let us continue with sieving. After a while, we may figurally notice that ‘lots of numbers get struck out more than once’. And if we look for what *constrain* our action, we may notice that in general when we are coping with a prime number p, its multiples by numbers smaller than p will already have been dealt with, and the first one that hasn’t been will be p times p (= p^2).

And this one in turn will facilitate our action. For example, ‘when we dealt with 2 and 3, leaving 5 as the next prime, the remaining numbers, 5, 7, 11, 13, 17, 19, 23, below 5^2 = 25 where therefore already known to be prime.’

It is important to notice that checking has been embedded in at the outset of sieving. So when we finish sieving, the remaining numbers are the prime numbers and we do not need to check the status of each individual number, provided that we are aware of what the sieve has been based on.

In general, when generating an example, two aspects are inseparably intertwined: the ongoing choices (the choice of the medium in which the example is going to be represented and/or the choice of certain universal sets) and the ongoing changes. A change has two different but interrelated aspects, *figural* aspect and *regulating* aspect. Figural aspects embrace all those changes in, and actions upon, structuring and restructuring intermediate objects that are supposedly more familiar, hence more concrete, than what the required example exemplifies. The required example is one of those structures. Intertwined with the figural aspects are regulating aspects, those things that constrain, control and drive figural changes. Among those that regulate the figural changes are the figures themselves, the given definition per se, the interpretation of the given definition, and the expectation of what an example should look like. It is worth stressing again that figural aspects and regulating aspects are intertwined. On the one hand, figural changes show us what regulate them, and on the other hand, they are sprung from and shaped by what regulate them.

Eventually, all these choices and changes will end up with something that is likely to be the required example. Here again we have two possibilities. First, our student is aware that the changes that he/she has made use of guarantees that the product is an example. Second, he/she is not aware that the changes guarantee that the product is an example. In the former, the checking stage is embedded in from the outset. In the latter, our student is apt to *check* the status of the product for being an example. In other words, generating is related to checking, one way or the other.

Let us now turn back to the present study. Since all students in this study were asked to present their examples using a given representation (grid) the choice of a medium was not a problem as it could be in other situations regarding generating an example. Furthermore, the given grid worked as the universal set in which the students generated their own examples and non-examples. Moreover, the given representation brought students into a more familiar context in which they could literally see, draw, and make their figural changes. As a result, the distinction that we have intended to make is based on what regulated their figural acts. We have recognized two general ways of generating an example, *figural generating and conceptual generating*. In the following, we will use students' work to exemplify these two ways of generating an example.

Figural Generating

When the students focused on the given representation in itself, they involved themselves in certain *figural processes* that could supposedly result in an example. The language that they used was a *figural language* that was confined to the medium in which they presented an example. If what they made happened to be an example, they were at best looking for certain *figural patterns* in those successful figural processes. However, more often than not, the processes that they made use of were not *predictive*, i.e. making use of them did not guarantee that the product would be an example.

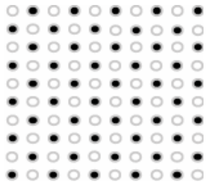
It is worth considering that due to this characteristic of figural processes (they are not predictive) *checking* the status of the product was inevitable. Even if they were looking for more examples (often at interviewer's request and less at their own will), they often found it hard to generate new examples.

Let us give some examples. Dick is a first year law student. On the interviewer's insistence he has already made one example.

> Interviewer: would you like to make another one?
>
> Dick: I do it again, yeah (he is clearly excited about "coming with another *system* that works"; as a result, he is willing to make another example)
>
> Dick: For this one (the first example), I worked from the top to the bottom…, now I am considering filling from left to right to the middle spots…

Soon afterwards he realised that "that would be resulted in the same sort as that (the first example)! But, still focusing on the figural features, he continued as follows:

> Dick: …if you do a system where you fill in blocks of dashes…something possibly diagonal lines, but I am not sure that system work, I must fill in to see it works…so that each circle, each visiting-city is not next to another visiting-city, um, so that, like that (laughing), I am not sure this works, because you have to have every, you have to have (pausing), I think this does work actually (laughing), I just took it and it worked!

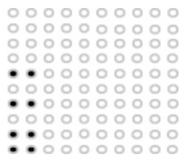


Conceptual Generating

One of the basic ways of conceptually generating an example is making use of the given definition as such (if the given definition allows doing so). Let us give an example.

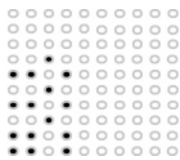
Amin is a first year mathematics student. He has already chosen the visiting-cities of city one:

When choosing the visiting-cities of city two, he realized that "because we have to have two, that means we have to have identical to one":

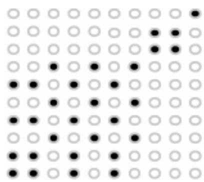


Then after "making three different to the others", when he came to the city four, he was only matching it up with the city two, and took the identicalness of four and one for granted. In other words, experiencing the transitivity of the matching procedure, he did not need to check all the possible pairs to see whether they are working according to the laws or not.

> Amin: for four we have to have four, if you take the pair two and four, because they both have this one in common they must be identical



Amin continued to match the pairs up until he ended up with the following figure while he announced 'leave it':



'Leave it.' There is no further need to check the above figure against the given definition to see whether it satisfies the required conditions or not; however, this needlessness to further check is tightly connected to (1) being aware that satisfying the defining properties or the necessary conditions when generating an example is equivalent to having an example (2) being aware that the defining properties have been satisfied. In Amin's case, the latter is highly connected to what he has taken for granted, i.e. the transitivity of the matching procedure.

It appears that in our thought experiment we have taken the situation to the extreme where there is no impression of the given definition as such, and there is no interaction between what has been defined and the other figural and conceptual entities. However, as Amin's use of transitivity shows, it is a simplistic view.

In general, a definition relates certain familiar concepts to each other in a particular way. But it is not the only way that those concepts could be related to each other and to the other concepts. When generating an example where concepts are tightly bound to objects, certain conceptual relationships could come to students' notice, could be examined against the background of the given definition and/or the other examples, and they could be accepted or rejected. This basic observation is what makes the ways of conceptually generating an example different from one time to the other, for one student, or from one student to the other.

CONCLUSION

Through the limitation on space, if not the limitations of our knowledge for the time being, we could only scratch the surface of a very complicated problem that surprisingly has been ignored in the literature. However, even now, we should add that the distinction between figural and conceptual generating is not intended for making a distinction between students. There is certain circularity between figural generating and conceptual generating, i.e. on one hand the figural generating could result in the conceptual generating, and on the other hand, as a result of, for example, competence, any conceptual generating could be used as it is a figural generating. Given this, it can be seen that the distinction made could not be a distinction between students.

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