# THINKING ALGEBRAICALLY ABOUT EARLY NUMBER

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*This paper describes an exploration of algebraic thinking about early number operations with students on our primary Initial Teacher Training (ITT) programmes. While the ITT curriculum expects students to engage with algebraic ideas that ultimately relate to early number concepts, the students themselves did not appear to link what they are learning to the teaching of Numeracy in schools. We therefore devised a teaching session, which worked with generalised arithmetic ideas in depth and assessed its effectiveness in helping the students make sense of the underlying mathematical ideas. This is very much 'work in progress', but early findings show that students are able to make the links, but that there are differences in students' understanding of the relationship between early number and algebra according to the type and stage of their training.*

## INTRODUCTION

The motivation for this work came from several directions at the same time. At the time one of us (AP) had been reading and reviewing a new book on children's arithmetic and algebra (Carpenter, Loef Franke, and Levi 2003). This shows some very exciting work with elementary school children in America who are working with their teachers on generalised arithmetic, which is later expressed in algebraic form. The book and related video CD show young children confidently finding generalisations in arithmetic, making conjectures, and writing some of these in algebraic form. What the book does not explain is how the researchers worked with the teachers in order to bring their mathematical understanding to a stage where they could confidently work with the children in a new way. I was unsure that teachers would be able to work in this way just by reading the book, given the evidence that we have about student teachers' and practising teachers' confidence in algebraic thinking, which raised the issue of the degree to which we addressed generalised arithmetic and algebra with our primary Initial Teacher Training (ITT) students. Could we explore the underlying algebraic structure of elementary arithmetic taught in schools with these students in order to produce better mathematics teachers?

At the same time BD had been reflecting on the teaching of such students. The Teacher Training Agency (TTA) requirements for the subject knowledge of primary trainee teachers, originally loosely based on the Key Stage 4 curriculum, made demands upon trainees' understanding of algebra that were difficult to reconcile with their needs as primary teachers, and even more difficult to achieve in the context of a teacher training programme, for example, factorisation of algebraic expressions. At the same time the National Numeracy Strategy (NNS; 1999) in speaking of *Laying the Foundations for Algebra* made it clear that the structural thinking about number and number operations that might be considered the essence of algebra, was embedded within the primary curriculum, including the use of letters to represent variables. The mismatch at that time was that the NNS requirements were wholly aimed at understanding number, while those of the TTA seemed to be algebra for the sake of it. More recent TTA guidance (2002) focuses on the mathematical knowledge required to teach all pupils effectively in the Key Stages for which the teachers are training. This has freed teacher educators to pay attention to the motivation that their student teachers may have for learning mathematics, that is, their ability to teach it well; and to consider what type of mathematical experience, both of content and process, will best enable the emerging teacher to acquire mathematical knowledge in a manner which will encourage its transformation from *learner knowledge* into *teacher knowledge* (Prestage and Perks 2001).

If teachers are to understand deeply the content of the curriculum they teach, we believe that their understanding needs to be *structural* and hence algebraic. It is the focus on structure that lies at the heart of algebra, rather than the use of letters. However, the ability to make connections between verbal and symbolic descriptions of a mathematical structure can strengthen understanding. As an example, the distributive law for multiplication over addition can be succinctly described by *a x (b + c) = a x b + a x c*. Being taught this as a fact will probably not help trainees to understand distributivity. If, however, they themselves can capture the essence of the way that they observe numbers to behave, through reflection on repeated experience, and can then seek to express the underlying structure as concisely as possible, codification of their generalisation using the language of algebra will, we believe, support their understanding.

Current ways in which we teach the use of letters as algebraic variables and unknowns are the use of the nth term to describe geometric and number sequences (a discrete valued function); the concept of a function acting on all numbers (a real valued function suitable for graphing); and the solution of simple equations, and we would wish to complement but not supplant these. Each of these is an extension of the algebraic ideas that are met within the primary curriculum. What we are now seeking to develop is a more reflective understanding of the arithmetic of the primary curriculum, based upon recognising its algebraic underpinning.

## WORK IN PROGRESS

In order to explore some of these ideas more fully with the students, we devised a teaching session for a group of student teachers who were all teaching assistants in primary schools, undertaking a four-year undergraduate teacher education degree. On this course the students spend three days of each week working as assistants in school, one day a week in the university, and one additional study day. The session was co-taught by both tutors and the students were asked at the end for feedback on what they had learnt. Our initial session was planned for students in the third year of this course, and as a result of their feedback, we also taught the session to students in the second and first years of the course. The indented transcripts included in our description of the session reflect the observations of a colleague who was asked to focus on the quality of student learning. He was present during the session with the first year students.

## THINKING ALGEBRAICALLY ABOUT EARLY NUMBER

The objectives for the session were:

- Recognise patterns underlying number operations
- Understand how using patterns in number can develop fluent calculation
- Understand how recognising a pattern can lead to a conjecture
- Become confident in using a symbol to represent any number (variable)

### Equivalence

The session then started with the understanding of equivalence. The students were offered: 7 + 3 = and asked for a solution. Having elicited 10 they were then asked for other whole number solutions. This generated discussion of both commutativity 7 + 3 = 3 + 7 and number pattern 0 + 10, 1 + 9, 2 + 8 etc.

Finally they were offered 10 = 7 + 3 and discussed what children might make of an equation written in this way. The use and meanings of the equals sign were then discussed.

### Adding zero

Next students were asked to solve, in quick succession:

7 + 0 = 59 + 0 = 79354 + 0 =

With the final one they were asked if they had ever seen that calculation before, and if not, how did they know the answer. They were then asked to work in groups to formulate a general statement, in words, about what we believe about adding zero.

> The students appeared to need more time to develop their thinking… before articulating the conjecture.

They were ultimately successful in explaining that adding a zero to any number left the number unchanged. The time required was not to understand the concept, but to apply the less familiar skill of putting it unambiguously into words. This led to the challenge to use a letter to represent any number, and represent the conjecture in symbols. Each group responded with 'n + 0 = n' or an equivalent statement, in some cases following considerable discussion to convince all members of the group. The time taken convinced us of the value of introducing symbols in the context of a generalisation that was so familiar to the students (adding zero). Familiarity can make a situation concrete even in the absence of a physical representation. The discussion was extended to adding zero to fractional and negative numbers, each group having, despite their use of a letter to represent numbers, confined their arguments to natural numbers. Again there was lively debate before general agreement was reached. The more abstract nature of negative numbers contributed to the length of the debate; fractions posed few problems. Students were then asked to produce a verbal and symbolic generalisation about subtracting zero.

> … when applied to subtraction there was a sense of growing confidence based on more discussion and a greater willingness to offer explanations and application. All the groups quickly arrived at n - 0 = n.

We discussed the relationship between verbal articulation and symbols, and the fact that different people had different ways of moving between them. Some found it easier to start with a symbolic statement, but agreed that verbalisation challenged their thinking. Others found symbols difficult, but could generate the symbols first passed through the verbal stage.

> The debate was opened up by the question of the problem children and adults have with symbolisation. This led to an excellent discussion and a general agreement, partly based on anecdotes from their own school experiences, that there still isn't enough discussion about maths including the role of symbols.

**Adding one**

The next stage began with 7 + 1 = and 8 + 1 = . It took some time to elicit an articulation that the answer was the next number in the counting sequence, and raised the question about whether, as teachers, we make explicit to children how to add one. Threlfall and Frobisher (ref) note that children who can add fluently use counting for adding one and two, and often use related procedures for larger numbers. The agreed conjecture in words was *'If you add one to any number it gives the next number in the counting sequence.'*

The observer noted:

> I interpreted this as a key moment in the session because all the groups were willing to attempt this task. One group symbolised this pattern immediately; most achieved it after a short time, while the level of discussion and collaboration was uniformly high. The most impressive elements were the willingness to speculate and then justify decisions. A lot of mathematical language also came out including ordinal numbers and the importance of counting in the early years and not always from 1.

The symbolic representation produced 'n + 1 = n + 1' which caused some dissatisfaction until somebody suggested 'n + 1 = (n + 1)'. The use of brackets to distinguish between n + 1 as an operation to be performed and (n + 1) as the resulting number proved a revelation to many students, who had only previously seen brackets as indicators of precedence within BODMAS. The students were then asked to consider subtracting one and multiplying and dividing by one.

> … they were all willing to engage in discussion, collaboration and experimentation. … I felt that the quality of the debate significantly increased particularly the relationship between language and symbolisation…

Students noted that in their group there were some who would go straight to the symbolisation while others preferred to work with words until they were satisfied with the articulation of their generalisation, and they could they translate this into symbols. A discussion arose about how there would be children like this in any class and teaching just one way, for example going straight to symbols - which had been their experience of secondary school mathematics teaching, would not aid all children's understanding.

**Using pattern to aid calculation**

Emboldened by their growing fluency with symbols students were asked to consider the three pairs of calculations:

| | | |
|---|---|---|
| 423 + 84 = 527 | 2128 + 340 = 2468 | 6582 + 647 = 7229 |
| 423 + 85= | 2128 + 341 = | 6582 + 648 = |

The students recognised this use of pattern to aid calculation as one taught in Key Stage Two. Again, verbal articulation of conjectures preceded symbolisation, e.g.

> *'When you add a number that is one bigger you get an answer that is one bigger'.*

We demanded more precision: 'Which number is one bigger? Would it matter?' and eventually arrived at

> *'The result of increasing by one either of a pair of numbers to be added, increases the total by one.'*

This led to the symbolisation $a + (b + 1) = (a + b) + 1$.

Somebody recognised this as the associative law, and another group noted that it did not only work for one, any number would work as well.

Students were next given the three pairs of calculations:

| | | |
|---|---|---|
| 294 – 29 = 625 | 6582 – 647 = 5935 | 672 – 85 = 587 |
| 294 – 28 = | 6582 – 646 = | 672 – 84 = |

Verbal conjectures were again elicited, and, after discussion, were of the form

> *'If you subtract a number that is one smaller, then the answer is one bigger.'*

This took some time, not so much because the phenomenon was difficult to articulate, but because many students needed reflection on why this occurred and to convince themselves that this result was inevitable. The symbolic generalisation $a - (b - 1) = (a - b) + 1$ caused a great deal of interested comment, with students relating it to the half-remembered 'two negatives make a positive' of their schooldays. Students also noted that subtraction was not associative.

## REFLECTIONS

As indicated above we taught this session first to the third year students on the part-time work based BA course. The students were unanimous in thinking that the session had been very valuable in helping them to understand a lot more both about algebra and about how it related to the arithmetic they saw taught in school. Several students said that this was the best mathematics session they had ever had and there was a real sense of them having put together bits of knowledge from their own schooling, the mathematics they had been learning in the university and their work with children, to really make mathematical sense of this arithmetic.

We explored with them the following two questions

1. Would you feel able to work with the children in your classes on this relationship between generalised arithmetic and algebra?

All the students could see areas of the numeracy curriculum that they had been involved in teaching which would lend itself to a more generalised approach, with children being asked to articulate conjectures and possibly writing it in symbols. There was a level of confidence in their ability to see possibilities for real change in their teaching in areas that they had not experienced previously in school.

2. When, on your four-year course, do you think that you would be best able to understand this session?

Most of the students thought that it would be best to learn these ideas earlier in their course. We therefore planned and taught a similar session to students on the second and first years of this course. While all the students expressed appreciation of the session, and were observed to be working at a high level of engagement with the mathematics, relating this to examples of children's learning in school we felt that the third year students brought more to the session from their previous mathematical learning at the university and experiences in school, indicating that such a session would be most valuable towards the end of an ITT course or perhaps as an in-service session for practicing teachers.

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