THE EXERCISE AS MATHEMATICAL OBJECT: DIMENSIONS OF POSSIBLE VARIATION IN PRACTICE

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By treating collections of questions as mathematical objects, that is ordered sets containing individual questions as elements, we gain insight into the potential role of exercises in learning mathematics. We use the notion of 'dimensions of possible variation', derived from Ference Marton, to discuss some exercises. There are implications for the design of question sets, for pedagogical decisions in the use of question sets, and for reflective questioning by learners.

ABOUT PRACTICE

The word 'practice' in the title is deliberately ambiguous. Teachers and textbook authors typically describe the use of repetitive exercises as providing 'practice' for learners, often without stating what such practice is supposed to achieve. Practice can mean the use of repetitive tasks to build up fluency, speed and accuracy in performing technical tasks. The importance of this in mathematics has long been recognized, from the use of Vedic sutras for arithmetic (Joseph 1991) through to Hewitt's search for economy in learning repeated actions (Hewitt 1996). However, many textbook exercises do not seem to offer practice of this type at all. Questions are more likely to be slightly different in a seemingly arbitrary way so that learners tend to proceed in a stop-start fashion.

Consider this selection of textbook questions on ratio:

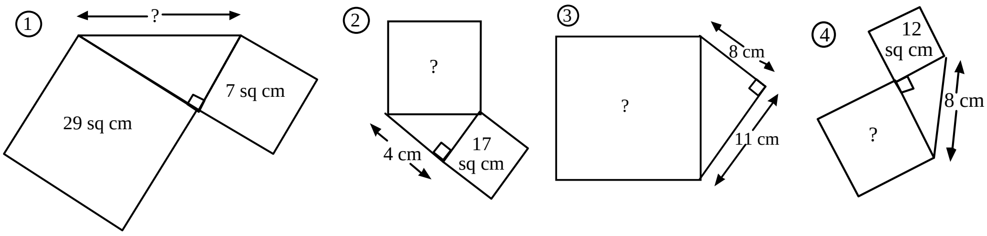
Reduce to simplest terms:

(a) $\frac{4}{12}$	(b) $\frac{36}{12}$	(c) $\frac{240}{300}$	(d) $5 : 5$
(e) $ab : ab$	(g) $2\frac{1}{4} : 1$	(h) $\frac{6ab}{3b}$	

The authors presumably hope that the learner will learn about a variety of forms of ratio, but the specific instruction is to express the ratios in simplest form. To get the answers the learner does not have to focus on notation, meaning and representation, merely on finding common factors to cancel. The exercise can be completed without any engagement with the ratio concept. It can also be completed without any increase in fluency because each question is different enough to need some fresh thinking. In classrooms, we observe that such exercises seem to result in a slowing down of pace and an increase in effort, rather than speeding up and reduced effort, unless teachers explicitly engage learners with the goal of getting faster and becoming 'experts'. Without some pedagogic intervention little can be achieved except for counting the right answers, and the analysis of errors to inform future teaching.

In contrast, the following exercise, designed by a teacher for his own use, provides some *systematic* change, each question varying in only one way from the one before.

Pythagoras Revisited: work out the missing areas or lengths in each of these:



Fluency is unlikely to be an outcome, but the learner has gradually been taken on a journey of conceptual development through being exposed to a variety of different forms, and thus explored many aspects of the typical Pythagoras' diagram.

The following is adapted from a larger exercise in Tuckey (1904):

Multiply each of the terms in the top row by each of the terms in the bottom row in pairs:

$x - 1$	x + 1	$x + 2$	$x + 3$
$x - 1$	x + 1	$x + 2$	$x + 3$

Apart from being a simple way to produce a long exercise, this appears to offer enough similarity to encourage fluency and some awareness of, and control over, change which may allow learners to get a sense of underlying structure while doing the examples.

DIMENSIONS OF VARIATION IN MATHEMATICAL STRUCTURE

In this paper we extend the application of Marton's work to mathematics education (see also Leung 2003). This paper is a small part of a body of work on designing tasks which engage learners in active construal of abstract mathematical structure. The learning theory and ontological position on which this work is based can be found in more detail elsewhere (e.g. Watson & Mason 2002, Mason & Johnston-Wilder 2004).

Marton's identification of 'dimensions of variation' offers a way to look at exercises in terms of what is available for the learner to notice (Marton, Runesson & Tsui 2004). This approach offers a structured and structural approach to exposing underlying mathematical form. We find it useful to consider dimensions of *possible* variation as experienced by a learner or by a teacher in any given situation (what could change and still the situation remains much the same), since this varies both between learners and even within one person at different times. In the case of 'Pythagoras Revisited' above, not only can individual numbers be changed, but lengths can be replaced by areas of squares on those lengths, and the quantity to be found can be related to the hypotenuse or to one of the other two sides.

By asking the highly mathematical question 'what changes and what stays the same?', and by examining the nature of the changes offered, we can be precise about what an exercise affords the learner. If too many dimensions are changing at once, and if the changes are not systematic, the learner is likely to overlook possible dimensions of variation in the effort to solve the individual problems, one after another, as one might do in the ratio exercise above. Learning takes place when something happens which is not ordinary, which perturbs learners' expectations, and which provides near-simultaneous differences to be discerned (Marton, Runesson & Tsui 2004).

ANALYSIS OF AN EXERCISE

The following exercise, taken from Krause (1986), appears to be of a typical 'do a few examples' kind but there is rather more sophistication than is first apparent. In his textbook, Krause does not say in advance what the exercise is about.

$Dt(P, A)$ is the shortest distance from P to A on a two-dimensional coordinate grid, using horizontal and vertical movement only. We call it the taxicab distance.

For this exercise $A = (-2,-1)$. Mark A on a coordinate grid. For each point P below calculate $Dt(P, A)$ and mark P on the grid:

(a) $P = (1, -1)$ (b) $P = (-2, -4)$ (c) $P = (-1, -3)$ (d) $P = (0, -2)$

(e) $P = (\frac{1}{2}, -1\frac{1}{2})$ (f) $P = (-1\frac{1}{2}, -3\frac{1}{2})$ (g) $P = (0, 0)$ (h) $P= (-2, 2)$

0To make sense of our comments it might be helpful to try the exercise for yourself.

We have used this task with about 120 people in all: inservice and preservice teachers from all phases and school students. Afterwards we ask them to report on their experiences in group discussion, keeping notes of what was said. Analysis of the way people report tackling the exercise is revealing. Some choose to plot all the points first and then calculate all the distances; others choose to calculate all the distances and then plot all the points; others do each point separately, finding the distance and plotting the point as Krause suggests. Whichever way they do the exercise, those who have not met this material before report remarkably similar experiences. They find themselves seduced into making generalizations early on in the exercise: that all the distances will be 3 and/or all the points are on a straight line. Most are not even aware they have made a generalization until the 'straight line' breaks down with the seventh point (g), which causes some surprise because it does not follow the expected pattern. This break in pattern causes many to begin to think about the mathematics behind what they are doing. They find themselves asking 'where would I expect points to be which are all a distance of 3 away from A?' or 'what has this straight line got to do with a distance of 3?' The main purpose of the exercise is, of course, not to get better at calculating distances and plotting points, as is clarified by the next question posed, which asks learners to make up more points P which have $Dt(P, A) = 3$, to graph them, and to describe the complete set. In our experience, most learners do this for themselves before they are asked to by the textbook.

Almost unanimously, people report that this exercise used their natural propensity to look for similarities and make conjectures in order to 'teach' them something about taxicab geometry; that they started by just 'doing' each separate point but were jolted into thinking mathematically by being offered points which broke their current sense of pattern, and that they had not realized they were aware of pattern until they were offered these points. Thus the combination of several similar examples and further not-quite-similar examples shifted them to work on a higher level than simply plotting and calculating, and this resulted in both improved fluency and conceptual learning as well as in motivation and interest.

A few people operated at a higher level throughout. As soon as they had done the second example they were already asking themselves 'why 3?' (seeing 3 as a dimension of possible variation, not simply as a fixed number). Some did not plot points at all, assuming this to be some visual extra which the 'teacher' required but which they did not themselves need; some of these proceeded to try to generalize algebraically. We have found very few other ways of responding to this exercise, and no examples of people who merely performed the required tasks without voluntarily conjecturing and predicting. It is possible, of course, that without our expectation that they would reflect on the task, fewer people would have brought their sense of pattern to articulation.

What intrigues us, given the very different backgrounds, mathematical knowledge, goals and social contexts of the groups with whom we have worked, is that there is an almost universal response to the exercise, and that this response hinges on the way the individual examples relate to each other. We therefore analysed the exercise to find out what variations were available to be discerned by the learner, and when. Our analysis would vary slightly according to the way someone chose to complete the exercise, so here we shall focus on the case in which a learner both calculates and plots each point in turn. We look at what aspects might be varied (dimensions of possible variation), how far they are varied (range of permissible change) throughout the exercise, and what is thus available for discernment.

Point A is fixed, hence the learner can focus on comparing relationships to A, rather than jumping about randomly. Later, A can be varied to generalize such relationships. Also point A is not trivial, so it might have some generic qualities, and some particularities which need to be distinguished.

The first three points P fix $Dt(P, A)$ as 3. In this case, the fixing of 3 is essential to the mathematical plan; the dimension of possible variation has been constrained to points whose distance from A is a constant. For some learners this may suggest 'circle', but most report that the answer '3' becomes an expectation and, eventually, their work shifts from 'calculating distance' to 'verifying that the distance is 3'. An elementary conjecture has emerged from the subconscious. The range of change used so far is restricted to integer coordinates, positive and negative. The fourth point is, for most people, a confirmation that the points are to be in a straight line, and provides a self-checking opportunity. The fifth point alters the range of permissible change from integer points to fractions, but within a context which provides ways to conjecture and test how these are interpreted. By this time, most people have decided that the answer they have to make is 3, and the points are in a straight line.

Then comes the point (0, 0). This is a familiar point, so not one which is expected to cause any difficulty in itself, but definitely one which is not on the straight line. It is not unusual for people to indicate surprise, puzzlement, or a sense of error. Until now, the two possible generalizations have been the distance of 3 and the position on a particular straight line; it has even been possible to conjecture a relationship. But Krause knows that the full range of permissible change for points which are a distance 3 away from A has not been offered, and hence people may have been tempted to make unwise or incomplete generalizations. The inclusion of (0, 0) makes it possible to discern difference between the shape of the locus and a straight line, yet the straight line conjecture allowed learners to cope with fractions and set up an expectation which allowed difference to be observed.

The final point gives more information about what shape might emerge, and Krause stops there, leaving space for learners to think and make up their minds about what else might happen. There is still the position of point *A* which can be varied, and the fixed distance, but the central generalization about the locus has already been made through the development of what, at first sight, were ‘practice’ examples.

Analysis of what is possible at each stage of the exercise matches almost exactly the reported experience of the majority of people to whom we have offered it, the few exceptions being people who have leapt to a level of abstraction much earlier than the exercise expects, or those who have restricted themselves to a purely algebraic approach. Each of these exceptions ended up with a similar understanding of taxicab ‘circles’ as others, albeit by a different route. The close relationship between the author’s choice of dimensions of variation and associated ranges of change, and the learning experiences of those doing the exercise, suggests to us that such an analysis would be a good way to judge the quality of published exercises in terms of their potential to promote conceptual learning.

WORKING ON EXERCISE AS MATHEMATICAL OBJECT

Apart from in cases of well-designed exercises, such as Krause’s, it is not usually enough to have subtle differences lying around waiting to be noticed, as generations of school students trapped at repetitive deskwork can testify. There will always be some learners who approach even tedious exercises with the intention of making sense of underlying structures, but they are rare. To make it more likely that learners will step back from the work they have done to look across it as a whole, to notice similarities, to compare the similarities they notice, to become aware of aspects *as* dimensions of possible variation, to express them as generalities, to express algorithms and methods in their own words, it would be generally necessary to use explicit requests to do so (e.g. Watson 2000, see also Mason & Johnston-Wilder 2004).

Developing reflective practices amongst learners requires more than setting them routine tasks to complete. By constructing tasks for learners which provoke conjecture and modification, learners are called upon to use and develop their natural powers of mathematical sense making. By using mathematically based prompts such as 'what is the same and what different about?', 'what is changing and what is not?' (see Watson & Mason 1998 for a collection) learners might become enculturated into a practice of using an exercise as a stimulus to learning and stimulation to exploration, rather than a 'task' to be completed as quickly and effortlessly as possible.

Our tentative conclusions at this stage of our work are:

- control of dimensions of variation and ranges of change is way to design powerful exercises which encourage learners to engage with mathematical structure;
- analysis of dimensions of variation can indicate the potential of exercises within particular settings and situations;

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