STUDENTS' EXPERIENCES OF 'EQUIVALENCE RELATIONS'

Amir H Asghari

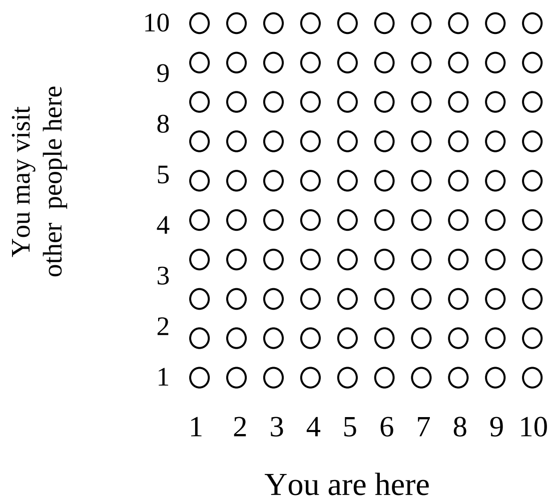
University of Warwick

We engaged a smallish sample of students in a designed situation based on equivalence relations (from an expert point of view). The students were different from each other in age and educational background, and all were unfamiliar with the formal treatment of equivalence relations. The study was conducted by holding individual in-depth task-based interviews, in which we aimed at investigating the ways that students organize the given situation, rather than teaching them any particular ways of organizing that. As result, I will report a certain way of organizing the given situation, from that a 'new' definition of equivalence relations, and consequently a new representation for them, is emerged; a definition that seems to be overlooked by the experts.

INTRODUCTION

Before giving any introduction in a normal way, let us invite you to give an example of a "visiting law" as defined below.

> A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.
>
> A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.
>
> A visiting law must obey two conditions to satisfy the mad dictator:
>
> 1. When you are in a particular city, you are allowed to visit other people in that city.
>
> 2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.
>
> The dictator asks different officials to come up with valid visiting laws, which obey both of these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below.

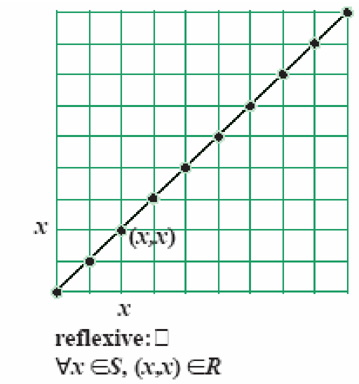


If you have not yet generated your own example, please before reading the next line that is about the original aim of this task try to generate one.

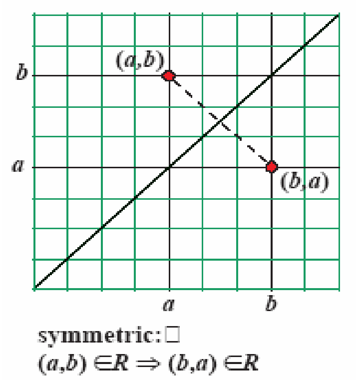
When devising this particular situation, the researcher had the standard formulation of 'equivalence relation' and 'partition' in mind (see below). And the situation was originally designed with the *intention* of seeing how the students proceed with what was then considered to be *the only way* of organizing the situation in order to come to *the* definitions of 'equivalence relation' and 'partition'.

Even though we can find different forms of the standard definition of equivalence relation in any text book about the foundation of mathematics (e.g. Stewart and Tall, 2000), let us choose one of them from a research paper that is highly related to the present study. Chin and Tall (2001) uses the following version of the standard definition of equivalence relation, i.e. a subset of S × S, say R, in which:

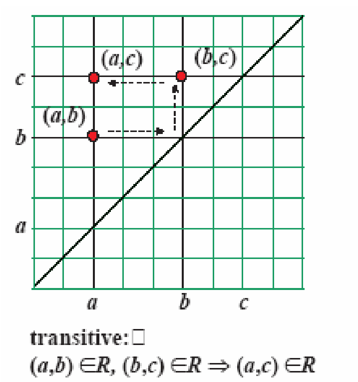
The elements (x, x) are all in R
(R is *reflexive*)



If (a, b) is in R then (b, a) is in R
(R is *symmetric*)



And it is *transitive*, i.e. If (a, b) and (b, c) are in R then (a, c) is in R. Although you could not find a picture for this form of expressing of the transitive property in the text books, Chin and Tall give the following picture for it:



If you generate an example of a visiting law and then try to generate more examples, you will wonder at the original aim of this study that was leading student from the above task to such complex definition of equivalence relation. Thus let us have a close look at the task and what we originally aimed for.

Analysis of the situation

As it can be seen an equivalence relation is first and foremost a relation. Thus let us start from relations in general. Set theoretic treatment of relations gives a unit and plural character to those elements that relate to each other. This aspect can be implicitly seen in the eloquent and still informal introductory paragraph of the chapter on relations in Stewart and Tall (2000, p.62):

> The notion of a relation is one that is found throughout mathematics and applies in many situations outside the subject as well. Examples involving numbers include ‘greater than’, ‘less than’, ‘divides’, ‘is not equal to’, examples from the realms of set theory include ‘is a subset of’, ‘belongs to’; examples from other areas include ‘is the brother of’, ‘is the son of’. What all these have in common is that they refer to two things and the first is either related to the second in the manner described, or not.

As it can be seen each one of that ‘two things’ in Stewart and tall examples implicitly belongs to a set; therefore, even though, for example, 1 in $2 > 1$ is treated as an individual, being in the set of integer gives an infinite access to it and illuminates its plurality. In general, those ‘two things’ are not only single individuals, but also something that can fill one of the two sides of a relationship, or more importantly fill both side of a relationship; they are simultaneously unit and plural.

As a particular relation, equivalence relation inherits above peculiarities in a more remarkable way. When we are looking for a concrete example of equivalence relation, we are apt to define a relation between *two* different things or people, say, *both* have the same colour, *both* live in the same street; we can check the possession of the given relationship between those two things or people by pointing to those two; even we can do that in a more concrete level, or using Dienes words(1976, p.9), in ‘first order attributes’ realms, say, they are both green, for the first relation, and they both live in Oxford street, for the second. However, as Dienes pointed out, the former way of checking, described by ‘second order attributes’, is more abstract and more difficult than the latter:

> To have the same colour as something else is a much more sophisticated judgement than to say that they are both green. (ibid, p.9)

Regardless of the difficulty, passing to ‘second order attributes’ realms seems inextricable for grasping reflexive property. To grasp reflexive property, first we must go one step further of the situation, and look at the situation as ‘…having the same colour as…’, ‘…living in the same street as…’, and so on; that demands, on the one hand, a transfer from unity to plurality in the sense described for relations in general, and on the other hand, a transfer from plurality to unity, i.e. coming from *both* to *each*.

In sum, although bringing plurality and unity together is hardly accessible in the concrete cases, we tried to achieve it, in the designed situation, by giving a “metonymical definition” in which more than often, city is used to refer to people in city. In consequence, “each city is its own visiting-city” metonymically stands for “in each city you can visit other people". And as it can be seen the former is an expression of the reflexive property. Having captured the reflexivity (the points on the diagonal), the situation aimed at leading students to the symmetry and transitivity through creating their own examples demanded in the first task and then giving the minimum amount of information demanded in the following task:

> The mad dictator decides that the officials are using too much ink in drawing up these laws. He decrees that, on each grid, the officials must give the least amount of information possible so that the dictator (who is an intelligent person and who knows the two rules) could deduce the whole of the official's visiting law. Looking at each of the examples you have created, what is the least amount of information you need to give to enable the dictator to deduce the whole of your visiting law.

Participants

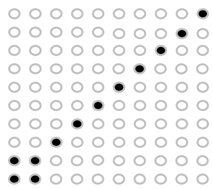
Having considered such details, our study started with a small opportunistic sample of students that their only commonality was that they had not been formally taught equivalence relations and related concepts. The initial data revealed that the students spontaneously create their own way of organizing the given situation which were not necessarily those intended by the situation designer; in other words they had their own concepts to use and their own ways of relating them to each other. Accordingly, the *intention* of the study became an investigation of *the ways* that students organize the given situation including a careful consideration of what *they* use to organize the situation.

Results

To manifest a flavour of the present study, let us present a snapshot of our data coming from interview with Tyler who is an undergraduate computer science student.

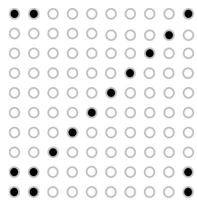
To satisfy the first condition of the given situation, Tyler blacked the diagonal and continued as follows:

Tyler: If I am in city one, and we allow to visit city two, how the other things need to change, to keep the rules consistent and see either they are completely the same or completely different, so aha, so city two now have to be able to visit city one…



Then he considers two things: "mirroring in y equals x"and "box"(square) and then "to see what was happening" he decides to make city one visit city ten:

Tyler: … and I realised first that, city ten has to visit city one… so that the second law …city ten has to visit city two…now I look at the city two, now I realised they are different from city one…so I copy number one on to number two also just to keep them the same…



As a result, Tyler abandons the "block square", keeps the "mirroring" and *proves* it as a "general pattern of these dots" (if (x, y) then (y, x)). In addition, the way that he proves "mirroring", gives him a new insight, i.e. considering the relationship between any two individual cities:

Tyler: If you allow a city to visit any other city, then it's gonna end up with having the same visiting-rules as that city that's allowed to visit and vice versa...

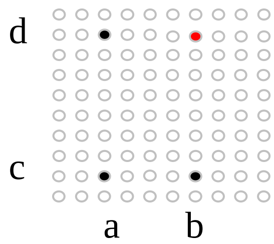
Having passed through many different concepts, he transcends the situation by introducing a new concept with general applicability (the 'box concept'):

Tyler: How do I say that columns must be the same mathematically? (He writes). If (x1, y1) and (x1, y2) and (x2, y1) then (x2, y2)

Interviewer: Could you explain.

Tyler: I think it's a mathematical way of saying …if a column has two dots, and there is another column with a dot in the same row, then that column must also have the second dot in the same row…I take maybe *a box of four dots*…I use the coordinate because that makes it very general, and so if I made that my second law, for a mathematician might be easier to follow.

It is worth saying that the box concept can be easily illustrated by a picture:

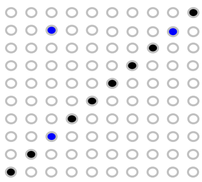


If (a, c) and (a, d) and (b, c) then (b, d)
(Box concept)

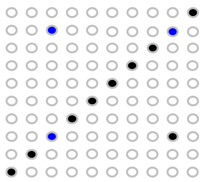
Given this, an equivalence relation can be understood as a relation having the reflexive property and the box property. That is, Tyler has explicitly generated a new (and, for us, unexpected) definition (which happens to be mathematically equivalent to the standard definition of equivalence) in order to organize this situation.

Equivalence relations, revisited

The following diagrams show how having reflexivity and box concept, we can deduce symmetry and transitivity.

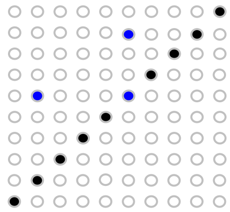


(**a, b**), (a, a), (b, b)
are three corner of the box

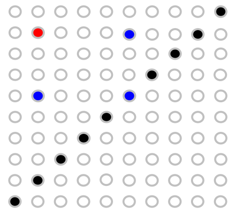


(**b, a**) is the
fourth corner

As it can be seen it is our old friend symmetry; the following diagrams illustrate the other one, transitivity.



(**a, b**), (b, b), (b, c) are three corner of the box



(**a, c**) is the fourth corner

On the other hand, it can be seen that having the normative definition of equivalence relation, based on reflexivity, symmetry and transitivity, we can deduce box concept.

Although the normative way of defining equivalence relations and its definition based on the box concept are logically equivalent, they have dramatically two different representations that could affect students' understanding of the subject. For example, Chin and Tall (ibid, p.5) suggested "*the complexity of the visual representation*" as to the transitive law as a source of a "*complete dichotomy between the notion of relation* (interpreted in terms of Cartesian coordinates) *represented by pictures and the notion of the equivalence relation which is not*". Accordingly, they suspected that that dichotomy inhibits students from grasping the notion of relation encompassing the notion of equivalence relation. However, the above figures show that the stated dichotomy, to a large extent, depends on the standard way of defining equivalence relation, i.e. if we define equivalence relation as a relation having the reflexive property and the box property, that dichotomy would disappear.

CONCLUSION

It is worth saying that the notion of equivalence relation defined by the box concept and its normative definition reveal two different ways of organizing the related concepts. While the former provides us with a simpler visual representation, the latter endows the subject with a seemingly more comprehensive quality in which two important types of relations, equivalence relations and order relations can be seen as particular types of transitive relations. Generally speaking, relinquishing a concept suitable for organizing a local situation in favour of grasping a more global picture appears as a particular aspect of mathematics.

REFERENCES

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