USING MULTIPLE REPRESENTATIONS TO ASSESS STUDENTS' UNDERSTANDING OF THE DERIVATIVE CONCEPT

Victor Amoah, Alperton Community School, Wembley

Paul Laridon, University of the Witwatersrand, Johannesburg

Calculus is highly symbolic in nature and therefore students often try to get through calculus by manipulating the symbols without understanding the meaning of such symbols (i.e. having a procedural but not a conceptual understanding of the topics in calculus). Educators are looking for ways to help students achieve higher levels of conceptual understanding. This study explored Science Foundation Year students' graphical, numerical and algebraic understanding of the derivative concepts after differential calculus course. The course was designed to develop students' conceptual understanding of the derivative concept.

INTRODUCTION

Tall (1996) points out that for students who take an initial calculus course based on elementary procedures, there is evidence that this may have an unforeseen limiting effect on their attitudes when they take a more rigorous course at a later stage. For example, Ferrini-Mundi and Gaudard (1992) found that it is possible that procedural, technique-oriented secondary school courses in calculus may predispose students to attend more to the procedural aspect of the college course. Students can be seen to be developing short-term techniques for survival when experiencing conceptual difficulties in the calculus at university. One of the possible approaches to teaching calculus Tall (1996) suggested involves numerical, symbolic and graphical representation.

A number of influential professional groups have put forth compelling proposals for the reform of mathematics education and calculus in particular. These proposals express a new vision of mathematical achievement, in which conceptual understanding plays a central role. According to Ohlsson (1987), one effect of developing conceptual understanding is that procedures and principles become easy to learn and understand. Conceptual knowledge is equated with connected networks of knowledge. In other words, conceptual knowledge is rich in relationships (Hiebert and Lefevre, 1986). On the other hand, procedural knowledge is defined as a sequence of actions. It is important to emphasize that both kinds of knowledge are required for mathematical expertise. Procedures allow mathematical task to be completed efficiently (Hiebert and Carpenter, 1992).

Assessment of multiple representation of derivative

The idea of connections between representations provides a way of thinking about assessing understanding and provides a general criterion for constructing useful tasks (Hiebert and Carpenter, 1992). Hiebert and Carpenter (1992) point out that errors may imply a lack of understanding, but an absence of errors on the type of items used on most diagnostic instruments does not imply understanding is present. They argue that the assumption that understanding implies well-connected knowledge suggests that we should assess understanding by attempting to determine how knowledge is connected.

Use of multiple representations, particularly when interconnections are formed, is expected to increase students' understanding. Some educators call for more emphasis on multiple representation of calculus concepts (Heid, 1988; Tall, 1996; Ostebee and Zorn, 1997) and this had made traditional skills test items no longer most appropriate for testing understanding. Therefore, a suitable assessment tool that reflects the teaching objectives should be developed. The primary purpose of this study was to assess the students' understanding of the derivative concept. This paper does not provide many details of the course itself but focuses on the aspect of assessing the understanding of the concept of the derivative.

METHOD

Key Questions

1. Does the student understand that, for a point a in the domain of a function f, the value of $f'(a)$ is the slope of the line tangent to the graph f at the point $(a, f(a))$?
2. In the absence of a defining equation for the function, is the student able to think about and work with the derivative using only graphical information?
3. Is the student able to think about and work with the derivative using numerical information?
4. Is the student able to apply the ideas of the derivative to solve a problem?
5. Is the student able to recognize the graph of a function if the graph of its derivative is drawn.

Participants

The participants for this study were 150 Science Foundation Programme (also known as **U**niversity of the **N**orth **F**oundation **Y**ear **P**rogramme – **UNIFY**) students of the University of the North in the Limpopo Province of South Africa. Of the 150 students, 24.5% were females and 75.5% males. The mean age of the participants in the study was 19.7 yrs (S.D. = 1.6). The students were selected from a pool of 802 students who wrote the UNIFY selection test at the beginning of the 2001 academic year. UNIFY is almost exclusively serving students from disadvantaged backgrounds. These are students who received their secondary education under adverse school conditions that could not provide them sufficient opportunities to realise their potential and thus gain immediate entry into mainstream courses in mathematics.

Procedure

This study took place in the year 2001. The data was collected after the UNIFY students had completed a course in differential calculus in the second semester of the 2001 academic year. A paper and pencil test developed by the researcher was administered to 150 students (from five different groups) who made themselves available for the test. The test was administered to students in their classroom. The teaching approach for all the five groups was similar in that the groups were taught with the emphasis on concepts. All five groups used the same worksheets. The worksheets were developed by UNIFY and contain mostly numerical, graphical and elementary applications of the derivative. One group in addition, used the graphing capabilities of computers in order to provide for the visualisation of the derivative concept. Teaching at UNIFY is aimed at mathematics sense-making. The teaching involved (i) continuous identification of student ideas which in this case was through the written work of students and through mathematical discussions that were held in class, (ii) continuous exchange of mathematical ideas between peers and/or with the facilitator/lecturer.

Test Items

The test was designed to obtain information on the students' conceptual understanding of differential calculus. The test items were initially reviewed by six experts in the fields of mathematics and mathematics education at the University of Transkei and the University of the Witwatersrand (in South Africa) for content validity. Appropriate modifications were then made and piloted in 2000 by administering the test to 160 science foundation year and first year students of the University of Transkei at the end of their calculus course. From the data generated by the pilot, further improvements wee made to the instrument.

The test items used in the main study and what they gauged are now discussed.

- The ability of students to calculate average rate of change graphically. (**Item 1a**)
- The ability of the students to find the derivative at a point from a graph. (**Item 1b**)
- The ability of the students to explain what is meant by a derivative at a point. (**Item 1c**)

Item 1

The sketch of the graph of a function was given.

a) what is the average rate of change from point A to point B?
b) Find the derivative of f(x) at x = 3?
c) What do you understand by the derivative of f at point A?

The average rate of change over an interval can be obtained from the *graph* by noting the amount of vertical increase (rise) or decrease (drop) in the function values as read from the graph over an interval of the independent variable and dividing this difference by the width of the interval. If the function is represented by a *graph*, students were expected to approximate the instantaneous rate of change of the dependent variable for a particular value of the independent variable by estimating the slope of the tangent to the graph at that particular point.

- The ability of the students to calculate the slope of a curve at a point algebraically. (**Item 2**)

Item 2

Find the slope of the curve at x = 1 for the function $f(x) = 2x^2 - x$.

The slope of the curve $f(x) = 2x^2 - x$ at $x = 1$ can be obtained from the equation by symbolically computing the first derivative and evaluating it for the appropriate value of the independent variable.

- The ability of the students of the students to estimate the derivative at a point numerically. (**Item 3**)

Item 3

The table below shows values of $f(x) = x^3$ *near* $x = 2$. *Use the table of values to estimate the derivative of f(x) at x = 2.*

x	*1.998*	*1.999*	*2.000*	*2.001*	*2.002*
$F(x) = x^3$	*7.976*	*7.988*	*8.000*	*8.012*	*8.024*

The derivative of $f(x) = x^3$ at $x = 2$ can be estimated from a *table of values* by computing the average rate of change in the dependent variable over intervals immediately before and after the value of the independent variable.

- The ability of the students to apply the derivative concept to solve a problem. (**Item 4**)

Item 4

When an antibiotic is introduced into a culture of bacteria, the number of bacteria present after t hours is given by $N(t) = 2000 + 10t - 5t^2$, *where N(t) is the number (in thousands) of bacteria present at the end of t hours.*

Find the rate of change in the number of bacteria present at the end of 2 hours.

Rate of change in the number of bacteria present can be obtained from the given equation by symbolically computing the first derivative and evaluating it for the appropriate value of the independent variable (i.e., at $t = 2$).

- The ability to recognize the graph of a function if its derivative is drawn. (**Item 5**)

Item 5

Students were given a diagram in which a particular curve was indicated as being the derivative of a function. Students then had to identify the graph of the function from a collection of curves in the diagram. . They were also required to explain how they arrived at their choice of curve for the function.

RESULTS AND DISCUSSIONS

Table 1: Results of Item 1a and 1b

Item	**Correct answer**		**Erroneous element**		**Incorrect answer**	
	n	**%**	**n**	**%**	**n**	**%**
1a	75	50.0	42	28	33	22
1b (*graphical*)	39	26.0	43	28.7	68	45.3

Item 1a was answered correctly by 50 % of the 150 students who wrote the test. 28 % of the students showed evidence of knowledge of the solution to the problem but minor errors occurred and 22 % of the students did not demonstrate knowledge of relevant procedure to answer this question.

In item 1b, graphical competency is demonstrated if the gradient of the tangent to the curve at the required value of x is determined. Most of the students could not find the derivative at a point from the graph. Only 39 (26 %) students out of the 150 students were able to demonstrate their ability to calculate the derivative at a point graphically. Some of the incorrect answers came about because some of the students confused the derivative at the point with y-value of the point of tangency. Other students endeavoured to find an equation for the function represented graphically. Some students made errors because they had difficulty in computing the gradient of the tangent to the curve although the principle essential to the solution was understood.

Table 2: Results of Item 1c (*explain*)

Correct Explanation		**Erroneous element**		**Incorrect Explanation**	
N	**%**	**n**	**%**	**n**	**%**
90	60.0	23	15.3	37	24.7

Results of item 1c indicate that 60 % of the students were able to demonstrate their understanding of derivative at a point.

Table 3: Results of Item 2 (*algebraically*)

Correct answer		**Erroneous element**		**Incorrect answer**	
N	%	n	%	n	%
81	54.0	14	9.3	55	36.7

About 54 % of the students demonstrated the ability to calculate the slope of a curve at a point algebraically. Some students (9 %) knew how to differentiate but some errors occurred due to incorrect differentiation or manipulation. About 37 % of the 150 students did not demonstrate any conceptual knowledge of the problem, some instead substituted $x = 1$ directly into the equation for the function.

Table 4: Results of Item 3 (*numerical*)

Correct answer		**Erroneous element**		**Incorrect answer**	
N	**%**	**n**	**%**	**n**	**%**
56	37.3	4	2.7	90	60.0

Item 3 was answered poorly. Only 8 (37%) of the students managed to cope with this problem. Most of the students (60 %) were unable to estimate the derivative at the point numerically. Incorrect answers occurred because most of the students responded to this question by differentiating symbolically although symbolic differentiation was not indicated.

Table 5: Results of Item 4 (*application*)

Item	Correct answer		Erroneous element		Incorrect answer	
	n	%	n	%	n	%
4	44	29.3	18	12.0	88	58.7

Item 4 was answered correctly by about 29 % of the students. About 59 % of the students could not apply the derivative concept to solve this problem. Most of the incorrect answers came about because the students attempted to substituted $t = 2$ directly into the equation for the function.

Table 6: Results of Item 5 (*derivative Function*)

Correct answer		Erroneous element		Incorrect answer	
n	%	n	%	n	%
13	8.7	51	34.0	86	57.3

Students were asked to identify the graph of a function given the graph of its derivative. They were then asked to explain how they arrived at their answer. This item was answered poorly. Most of the students (57 %) did not identify the graph of the function correctly from the graph of its derivative. About 9 % of the 150 students managed to get the correct answer and provide an acceptable explanation. However 34 % could not explain their choice although it was correct.

Although calculus reform movement emphasizes the ability to move freely amongst multiple representations as central to building the interconnectedness which indicates understanding, the results indicate that there is not much consistency across the responses to various items. Despite the similarities among Items 1b, 2, 3 and 4 the students lacked the ability to move comfortably among the different representational modes as in symbolic equations, tables of values and graphs.

REFERENCES

Ferrini-Mundy, J. and Gaudard, M. (1992). Secondary school calculus: preparation or pitfall in the study of college calculus. *Journal for Research in Mathematics Education* 23 (1), 56-71.

Heid, M. K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. Journal for Research in Mathematics Education, 19 (1), 3-25.

Hiebert, J. & Carpenter, T. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook on research on mathematics teaching and learning* (pp.85-97). MacMillan Publishing. New York.

Hiebert, J. and Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp1-27). Hillsdale, NJ: Erlbaum

Ohlsson, S. (1987). Sense and reference in the design of interactive illustrations for rational numbers. In R. W. Lawler and M. Yazdani (Eds.), *Artificial intelligence and education*, Vol. 1 (pp. 307-344). Norwood, NJ: Ablex.

Ostebee, A. and Zorn, P. (1997). *Calculus from graphical, numerical, and symbolical points of view*. Harcourt Brace College Publishers. Orlando, Florida.

Tall, D. (1996). Calculus and functions. In A. Bishop et al.(Eds.), *International handbook of mathematics education*, Part 1(pp.289-324). Kluwer Academic Publishers. Dordrecht.