LIMITS - A SECONDARY SCHOOL VIEW

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This paper starts by examining the central part of the definition of a convergent sequence, namely, the use of a pair of inequalities to identify a unique number. This use will be called the 'carpenter's vice'. The secondary school curriculum (including single A level mathematics) is surveyed for those parts in which a limiting process is either explicit or implicit. Signs of the 'carpenter's vice' are sought. While the 'vice' occurs in school mathematics, it is never the centre of attention. This 'vice' is not part of the mathematical tool-kit of a typical beginning undergraduate.

THE 'CARPENTER'S VICE'

The standard definition of a sequence (a_n) converging to a number a is:

given $\varepsilon > 0$, there exists an N such that $n > N$ implies $|a_n - a| < \varepsilon$.

At the heart of this definition is the inequality $|a_n - a| < \varepsilon$, which, if we avoid the modulus notation introduced by Weierstrass in 1859, can be read as

$$-\varepsilon < a_n - a < \varepsilon.$$

To understand the significance of these two inequalities, ask yourself what numbers A might satisfy

$$-\varepsilon < A < \varepsilon$$

for all positive numbers ε. Can you find one number A? Can you find more than one number A? If you cannot find more than one A, think of a reason why there is only one such number.

When you are confident about your answers, ask yourself what numbers A might satisfy

$$-\varepsilon < A - 6 < \varepsilon$$

for all positive numbers ε. The two inequalities squeeze the number between them into a gap of width 2ε. Because 2ε may be arbitrarily small, $A = 6$ is the only possibility, and we refer to this process for identifying

$A = 6$ as the 'carpenter's vice'.

The justification of the 'carpenter's vice' appears as Lemma 1 in Newton's *Principia* (1687).

> Quantities, and the ratio of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.

> If you deny it, suppose them to be ultimately unequal, and let *D* be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference *D*; which is contrary to the supposition.

We now search of signs of the 'carpenter's vice' in school.

Limiting processes in the curriculum 11 - 16

1. Recurring decimals. The first place where a convergent sequence is implicit in the mathematical curriculum is that of recurring decimals. The result 1/3 = 0.333... may appear on a calculator or from the repeated division

$$\begin{array}{r|l} & 0.\quad 3\quad 3\quad 3\quad 3\quad ..\quad .. \\ \hline 3 & 1.\quad 0\quad {}^{1}0\quad {}^{1}0\quad {}^{1}0\quad ..\quad .. \end{array}$$

The changing significance of the remainders is not noted. The result is treated as a complete number, and checked by working out $10x - x$.

The sequence $\frac{1}{3} - 0.\underbrace{333..3}_{n\,digits} = \frac{1}{3 \cdot 10^n}$ is not explored.

2. Area of a circle. The second place where convergence may be detected in the school curriculum is in justifying the product

area of a circle = (radius) × (½ circumference).

The circle is dissected into an even number of congruent sectors which are fitted together to make a figure roughly resembling a parallelogram. As the number of sectors increases, the figure tends to a rectangle. It is an unusual example of convergence in that the area being examined remains constant throughout. The average height of the figure is greater than the radius; the average width is less than half the circumference. The limit is not examined numerically or with geometric inequalities.

3. π The third place where convergence is implicit is in obtaining the value of π. Pragmatic measurements may be made of the circumference of a circle using a thread which is wound around a cotton reel or by rolling a trundle wheel. Pragmatic measurements of area made be made by drawing a circle with a pair of compasses on squared paper. Precise bounds on the circumference may be made by claiming the circumference of a circle lies between that of an inscribed regular hexagon and a circumscribed square. This gives $6r < 2\pi r < 8r$. Precise bounds on the area of a circle may be made by claiming that the area of a circle lies between that of an inscribed regular dodecagon and a circumscribed square. This gives $3r^2 < \pi r^2 < 4r^2$. There is an incipient 'carpenter's vice', but it is left wide open! Archimedes' bounds of $(3\tfrac{10}{71})r^2 < \pi r^2 < (3\tfrac{1}{7})r^2$ are much tighter.

4. Volume of pyramid and cone Pragmatic measurements may be made with sand in a paper cone or water in a plastic pyramid. Precise calculation of these volumes is not attempted before calculus is available. A precise calculation can be made without calculus, using a 'carpenter's vice', when the formula for $\sum_{r=1}^{n} r^2$ is available.

Limiting processes in the curriculum 16 -18 (single A level)

5. Sum of a geometric progression. After calculating the finite sum

$$a + ar + ar^2 + ... + ar^{n-1} = S_n = a\cdot\frac{1-r^n}{1-r}$$

the case $r = ½$ is examined and then the texts claim that for $|r| < 1$, $r^n \to 0$, without further justification. This is the basis for deducing that for $|r| < 1$,

$$a + ar + ar^2 + ... = \frac{a}{1-r}.$$

Although it is possible to justify this with a 'carpenter's vice', this is not done in school texts. In at least one course (AEB), the result is applied to the finding of rational equivalents to recurring decimals.

6. Differential coefficient Almost every course gives the argument:

$$y = x^2$$
$$y + \delta y = (x + \delta x)^2$$
$$\delta y = 2x\cdot\delta x + (\delta x)^2$$
$$\frac{\delta y}{\delta x} = 2x + \delta x$$
$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 2x$$

Although it is possible to express this argument using a 'carpenter's vice', this is not done in school texts and, as it stands, the argument is vulnerable to Bishop Berkeley's famous criticism.

Exactly the same (Bishop Berkeley) criticism may be made of

$$\lim_{n\to\infty}\frac{12n+30}{n} = \lim_{n\to\infty}\left(12+\frac{30}{n}\right) = 12$$

where the algebra is performed for legitimate numbers, n, and then an illegitimate "number" is substituted for n. Again it is possible to justify such results with a 'carpenter's vice'. This is not done in school texts.

7. Fundamental theorem of calculus In most courses, a monotonic function is presumed, and a 'vice' $y\cdot\delta x < \delta A < (y + \delta y)\cdot\delta x$ is constructed which gives $y < \frac{\delta A}{\delta x} < y + \delta y$. With continuity, $\frac{dA}{dx} = y$ follows. The argument is forgotten as the classroom emphasis shifts to anti-differentiation.

8. Repeated bisection. Approximation to a root of $f(x) = 0$ is made by locating a and b such that $f(a) < 0$ and $f(b) > 0$. The value of $f(½(a + b))$ is calculated and the process repeated on either $[a, ½(a + b)]$ or $[½(a + b), b]$, whichever interval has the change of sign. This is an excellent example of a 'vice', originating with Bolzano (1817), which has significant applications in more advanced work. However, the process is slower

than either of the approximations 9 and 10, below, so it is mentioned and then ignored in school.

9. Fixed point iteration When the equation $f(x) = 0$ is rewritten in the form $x = g(x)$, and the sequences $x_{n+1} = g(x_n)$ investigated, a convergent sequence tends to a solution of $f(x) = 0$ provided g is continuous. Different choices of x_1 may lead to both convergent and divergent sequences and to convergent sequences with different limits. The distinction between these cases is illustrated graphically.

When $0 \le g'(x) < 1$, convergence occurs with a 'staircase'.

When $-1 < g'(x) < 0$, convergence occurs with a 'web', and in such a case convergence is not monotonic. A rigorous proof requires the Mean Value Theorem. Only one course (MEI) attempts an algebraic justification and this is done without an effective 'vice'.

10. Newton-Raphson Sequences constructed with $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ converge to a root of $f(x) = 0$, provided $f'(x) \neq 0$. The convergence is typically justified with a graphical illustration of f with a tangent. One course [Sadler and Thorning] offer an algebraic justification without a 'vice'. A rigorous justification can be made if this is treated as a special case of fixed point iteration.

CONCLUSIONS

In school courses, a 'vice' is always explicit in example 8 above and usually in example 7. In neither case does the 'vice' remain the focus of attention. An open 'vice' is indicated in example 3. Where a 'vice' occurs in school it is treated as obviously effective. Students in school do not become proficient in constructing 'vices'. The 'vice' is not developed as a proof-technique in school, which is what it becomes in undergraduate analysis.

Research on convergence has generally focused on student intuitions about sequences, or upon the finished product of the limit definition. One of the general findings is that students expect convergent sequences to be monotonic. This is contradicted in the case of geometric progressions (example 5) when the common ratio is negative, and in the case of fixed point iteration (example 9) with $g'(x) < 0$.

My belief is that if pedagogical progress is to be made in lessening the 'epistemological obstacle' of the limit definition, the route will be through experiences of the 'vice' in proofs.

A level courses scrutinised

AEB Mathematics for AS and A level
Cross, T., Middle, J., Mallon, W.: 1992, *Foundation Mathematics*, Heinemann Education.

Bostock, L. and Chandler, S.: 3rd edn. 2000, *Core Mathematics*, Nelson Thornes.

MEI Structured Mathematics
Hanrahan, V., Porkess, R., Secker, P.: 2nd ed. 2000, *Pure Mathematics 1*, Hodder and Stoughton.

Hanrahan, V., Secker, P., Porkess, R.: 2nd ed. 2000, *Pure Mathematics 2*, Hodder and Stoughton.

West, E.: 2nd ed. 2000, *Numerical Analysis*, Hodder and Stoughton.

Neil, H. and Quadling, D.: 2000, *Pure Mathematics 1 & 2*. Cambridge University Press.

Sadler, A.J. and Thorning, D.W.S.: 1987, *Understanding Pure Mathematics*, Oxford University Press, reprinted 2003.

REFERENCES FOR RESEARCH ON CONVERGENCE

Davis, R.B. and Vinner, S.: 1986, “The notion of limit: some seemingly unavoidable misconceptional stages”, *Journal of Mathematical Behaviour*, **5**(3), 281-303.

Sierpinska, A.: 1990, “Some remarks on understanding in mathematics”, *For the Learning of Mathematics*, **10**(3), 24 - 36.

Tall, D.O. and Vinner, S.: 1981, “Concept image and concept definition in mathematics with particular reference to limits and continuity”, *Educational Studies in Mathematics*, **12**, 151 - 169.

HISTORICAL REFERENCE

Fauvel, J. and Gray, J. eds.: 1987, *The History of Mathematics, a Reader*, Open University. [For Newton, see page 391. For Berkeley, see page 556.]