SOME ISSUES REGARDING FORMAL ALGEBRAIC NOTATION

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We examine case studies (in progress) of teachers engaging with reflective practice on argumentation. The data comes from three sources i) audio taped meetings where all the teachers share their work with their colleagues, ii) videotaped lessons and iii) interviews of the teacher after each lesson. The aim is to identify the connection between the teachers' stated strategies and intentions (or 'espoused theories') and their 'theories-in-use' (what they actually do). The interviews and the group meetings enable the teachers to reflect on their practice and discuss their 'espoused theories' whereas the video taped lessons give us the opportunity to search for instances of their 'theories-in-use'. In this paper data from one case study will be presented and discussed.

INTRODUCTION

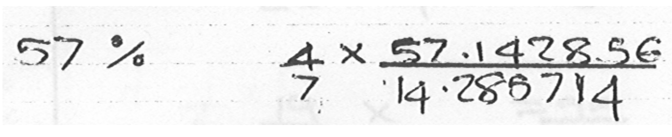
The literature on students' difficulties with algebra is quite extensive (e.g. Herscovics, 1989; Collis, 1975; Kuchemann, 1981). Even students who are relatively successful at mathematics and go on to higher mathematical studies are reported to have problems with the accuracy and fluency of their algebra (London Mathematical Society, Institute of Mathematics and its Applications, Royal Statistical Society, 1995). This paper explores the role that the visual impact of formal notation may have on some of these difficulties. Elsewhere (Hewitt, 2001) I suggest that some of the difficulties students experience with algebra might be as much concerned with notational matters as algebraic matters, and as such it is important to clarify the difference between these two. For the purpose of this paper I will consider algebraic activity to involve three aspects (Kieran, 1996): generational activities, transformational and global, meta-level, activities. The last of these involves such activities as problem solving and finding structure. Kieran stated that "such activity could be engaged in without using any algebra at all" (p. 272). By this I take her to mean without using any *algebraic notation* at all, otherwise there would be a paradox of defining an aspect of algebraic activity which could be carried out without algebraic activity! Thus, this third aspect of algebraic activity is independent of the presence of notation, whether formal algebraic notation or idiosyncratic personal symbolisation. Although listed third by Kieran, I suggest this type of activity is a pre-requisite to the first type of algebraic activity - generational activity. This first type is involved with the generating of expressions and equations but such expressions are expressions of insight or awareness. This requires a student to have engaged already in meta-level activity otherwise such insight would not be present and there would be nothing to express. A further point I wish to make is that the expression or equation generated does not have to be written in formal algebraic notation. Indeed students often express their awareness with their own devised symbolisation (for example, Hughes, 1990). Thus formal algebraic activity is not a necessary requirement for students to engage successfully in generational algebraic activity.

The second of Kieran's types of algebraic activity, and last that I consider, is transformational activity. This involves transforming existing expressions or equations, such as simplifying expressions or solving equations. Again, although such equations are sometimes understood to be written in formal algebraic notation, students can use their own personal symbolisation and get involved with the algebraic activity of simplification or finding a solution through manipulation. Thus I argue that all of Kieran's types of algebraic activity can take place independent of the presence of formal algebraic notation. The importance of this statement for me is not to downgrade formal notation but to realise its place and its relationship with algebraic activity. It is possible for students to work algebraically independent of formal notation. It is also powerful for students to be able to work algebraically with formal notation - to be able to generate it and to be able to transform it. What is of pedagogic significance is that the task of helping students adopt and work comfortably with formal algebraic notation is distinctly different to that of developing students algebraically. I suggest that students already can work algebraically, the issue is whether they can work algebraically with formal notation. Any difficulty doing so may be down to the adoption of formal notation rather than an ability to work algebraically. When formal notation is adopted, expressions or equations do not have to involve a letter. So, when I refer to written algebraic notation whether formal or otherwise, it does not imply the presence of a letter.

To explore the effect that notation can have on mathematical activity, I will offer some examples and activities in order to raise awareness of issues, and make reference to a variety of sources of which I give details as I proceed.

SOME EXAMPLES OF NOTATIONAL ISSUES

The first of three examples I offer concerns use of non-standard notation by a student who carries out correct arithmetic work. This highlights the fact that notational issues are a separate concern to algebraic or arithmetic issues. A teacher, Martin Duke, brought to my attention some work one of his students who was converting a fraction to a percentage:



The reader might like to try to interpret this work before reading on.

The bottom of the right-hand fraction is the result of finding one seventh as a percentage and the top is that multiplied by four. This gives the answer of approximately 57%. The multiplication is possibly a left-over from previous questions where the denominator of the original fraction was a factor of 100 and so the fraction was multiplied top and bottom by the appropriate number. The algebraic and arithmetic work is correct, it is only the notational aspect which might cause some difficulty for someone interpreting what had happened. As Richard Barwell (private conversation) pointed out it is not just a notational issue, but the relationship individuals have with notation. For a teacher or marker of scripts, there is the possibility of being unable to engage with the mathematics carried out by a student due to unfamiliarity with the individual notation used, and likewise a student may have difficulty with a task due to unfamiliarity with conventional notation rather than difficulty with any inherent mathematics.

The second example concerns some of my two-year PGCE students to whom I teach mathematics in the first year of the course. This highlights how notation can affect the correctness of the mathematics carried out. The students were working on a task where they were given the graphs of $y = e^x$ and $y = e^{-x}$ and asked to sketch on paper various graphs involving hyperbolic functions. One such graph was $y = \frac{1}{\sinh x}$. Initially, they drew the graph of $y = \sinh x$ and then made a notational 'error' of writing $y = \frac{1}{\sinh x}$ as $y = \sinh^{-1} x$ rather than $y = (\sinh x)^{-1}$. The $y = \sinh^{-1} x$ notation appeared to trigger previous work where the students reflected a function in the line $y = x$ in order to get its inverse. They carried out this reflection to produce the inverse graph of $y = \sinh x$ rather than its reciprocal.

The third and final example I offer concerns two equations which 40 mathematics teachers were asked to solve (amongst other tasks) within a questionnaire. This highlights how the visual impact of notation can affect the way we work with algebraic statements. The two equations were: $\frac{13}{2kx} = 47$ and $13 - (2 + k + x) = 47$. The reader might like to solve each of these equations before reading further, writing out any ‘workings’ along the way - just as the teachers had been requested to do.

These equations are structurally similar with there being a swap of operations: division with subtraction, and multiplication with addition. As a consequence of the similar structure, ways of solving the equations are also structurally similar. What I was interested in was whether the first written ‘step’ would involve the separation of *x* from the *k* and the ‘2’. Eleven out of the 40 teachers separated the *x* in this way in the first step of their working when solving $\frac{13}{2kx} = 47$. However, no teacher separated the *x* from the *k* and ‘2’ when writing the first step of the equation $13 - (2 + k + x) = 47$. There is no algebraic structural reason for this difference as both equations are structurally similar. However, there is a visual difference between the two notational expressions of the equations. With the division and multiplication, the notation does not ‘contain’ the *x*. Whereas with subtraction and addition, the need for brackets means the x is contained within the set of brackets and so is visually held with the k and '2' until a manipulation of the equation gets rid of the brackets.

READING FORMAL NOTATION

Sentences are read left-to-right for English speakers. In preliminary results from a study of Year 7 students from three schools (one selective girls grammar, two inner city comprehensives), several 'errors' made by students can be accounted for by the students having strict left-to-right readings of formally written arithmetic statements. Students were given a sheet of 17 arithmetic equations written in formal notation. Some equations were correct and some incorrect. Students were asked to place a tick next to those they considered correct and a cross next to those they considered incorrect. The three equations, which most students' answers differed from the conventional reading of the equations, were: $1+3\times 2=8$, $4+2\times 3=10$, and $2(3+2)=10$, with 40 (out of 79), 32 and 25 (out of 78) students respectively giving answers which would be consistent with strict left-to-right readings rather than conventional order of operations. However, there were equations where nearly all students had a different reading to that of strictly left-to-right. This was where an operation was carried out on the right-hand-side of the equation, such as $4+3=7-2$ where only two out of 76 thought this statement was correct. There were four other equations with operations on the right-hand-side (ignoring those involving division where other issues arose and are not dealt with in this paper) and these were also answered almost universally in line with conventional reading (only two non-conventional answers out of 313 responses in total for the other four equations). The '=' sign offers a potential break in a strict left-to-right reading, with a new beginning occurring after the symbol. Continuation of this study will involve consideration of whether equations which involve operations on both sides of the '=' sign can help students consider the possibility of a break in reading other equations which conventionally involve a non strict left-to-right reading on just one side of the equation. If so, equations with operations on both sides of the '=' sign may have this additional benefit as well as that of helping students develop a relational meaning for the '=' sign (i.e. *same on both sides*) rather than an operational meaning (i.e. *give me the answer to the statement on the left-hand side*) as reported, for example, by Booth (1988).

READING WORD STATEMENTS

Expressing non left-to-right order in written words can be problematic as well since words do not possess a set of notational conventions, such as brackets, to help with such readings. Instead phrases need to be added to indicate that a sequence is to be carried out operationally in a different order to that of the written text. During the session at the BSRLM Day Conference I asked people to participate in a variation of Chinese Whispers where each person had an algebraic statement (unique to them) written in formal algebraic notation at the top of the sheet. The task was to write the same algebraic statement underneath in words (the word 'bracket' was not allowed).

The original formal notational statement was then folded over so that it could not be seen and the sheet was passed on to the next person, who had to read the word statement and write it in formal algebraic notation underneath. The written word statement was then folded over and the sheet passed on to the next person. This continued with different people writing the ‘same’ equations in words, then symbolic notation, then words, etc. An example below indicates the difficulty involved in expressing when a break takes place. Here the use of the word ‘then’ is used in an attempt to indicate a break from the previous phrase ‘the result of’. This, however, had not been interpreted in the same way by the next person who included the ‘3’ within the square root.

$$2\sqrt{4+l}+3=5\times\frac{l}{7}$$

double the square root of the result of four add l then add three then equals five multiplied by l divided by 7

$$2\sqrt{(4+l+3)} = \frac{5l}{7}$$

SUMMARY

Notation can shape the way in which algebraic equations are manipulated as well as interpreted, and is of greater significance than just an issue of communication.

It is difficult to express order other than strict left-to-right with written words as there are no conventional phrases which indicate so clearly the beginning and end of ‘chunks’ as such notations as brackets, the root sign or the division sign manage to do within formal notation. There are some phrases such as ‘the result of’ which attempt to break the left-to-right reading and indicate a new beginning. However, these do not indicate an ending to the new beginning and so fail clearly to define the beginning and end of a ‘chunk’. In fact such phrases can bring their own complications. I gave the same Year 7 students mentioned above a sheet of arithmetic word statements, some of which were correct, some of which were wrong, and some of which were ambiguous. The students were asked to say whether they felt each statement was correct or not, and then express the statement in a formal notational form. I used the phrase ‘the result of’ within these sheets to try to indicate a break in the left-to-right reading. However, students sometimes responded to the phrase by actually performed a calculation, such as writing $\frac{6}{3}=3$ rather than $\frac{6}{1+2}=3$.

Non left-to-right ordering with formal notation also requires a break to take place and a new beginning to be formed. The use of the ‘=’ sign might help with this since most students seemed to read equations such as $4+3=7-2$ as incorrect. This requires a reading which involves a new beginning after the ‘=’ sign. If more students were accustomed to seeing equations with operations on both sides of the ‘=’ sign, then they would become also become accustomed to creating a break in the left-to-right reading of an equation. I speculate that this might help non left-to-right readings of equations where such a reading is required on just one side of the equation.

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