DESCRIPTIONS AND DEFINITIONS IN THE DEVELOPMENT OF THE CONCEPT IMAGE OF DERIVATIVE

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In this paper, we review the notion of Theoretical-Computational Conflict, discussed in Giraldo, Carvalho and Tall (2002). We intend to resume the discussion about the role of computer's limitations for learners' concept image of derivative and limit, from a broader theoretical standpoint. Our argument will be supported by a case study with a first year undergraduate student in Brazil.

INTRODUCTION

In Giraldo (2001), we defined *theoretical-computational conflict* as any pedagogical situation with apparent contradiction between the mathematical theory and a computational representation of a given concept. We have argued that the approach to the concept of derivative can be suitably designed to prompt a positive *conversion* of computational environments' inherent limitations to enrich, rather than narrow, learners' concept images (Giraldo and Carvalho, 2002; Giraldo, Carvalho and Tall, 2002). In this paper, we distinguish between the formal concept *definition and* the *descriptions* of the concept, that individually involve limitations and therefore do not fully reflect the mathematical definition. We will argue that suitable use of these limitations can stimulate students to engage in potentially enriching reasoning.

THEORETICAL FRAMEWORK

Our theoretical position is grounded in the theory of concept image and concept definition (Tall and Vinner, 1981). The *concept image* is the total cognitive structure associated with a mathematical concept in an individual's mind. It is continually being (re-)constructed as the individual matures and may (or may not) be associated with the *concept definition* (the statement used to specify the concept). Barnard and Tall (1997) introduced the term *cognitive unit* for a chunk of the concept image on which an individual focuses attention at a given time. Cognitive units may be symbols, representations or any other aspects related to the concept. A rich concept image should include, not only the formal definition, but many linkages within and between cognitive units.

In a strictly formal standpoint within a formal system of rules of inference, a mathematical object is perfectly characterized by its definition, so that the definition completely exhausts the object and, in this sense, a mathematical object *is* its definition. However, the theory of concept image suggests that the teaching of a mathematical concept must include different approaches and representations to enable learners to build up multiple and flexible connections between cognitive units.

DESCRIPTIONS AND DEFINITIONS: THE CASE OF THE DERIVATIVE

We will use the term **description** as any reference to a mathematical concept, employed in a pedagogical context, comprising inherent limitations with respect to the associated formal definition.

Each description of a given concept lays stress on certain aspects, but also casts shadows over others. The literature reveals examples of the *narrowing effect* (as discussed in Giraldo et al, 2002) of the students concept image as a result of focusing only on certain aspects, particularly computational ones. For example, Monaghan et al (1993) reported that students using Derive to study calculus explained the meaning of the expression $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ by replacing $f(x)$ with a polynomial and referring to the sequence of key strokes to calculate the limit.

One of the most widely used descriptions for the derivative in elementary calculus is the following: *The gradient of the function f at* x_0 *is the slope of the tangent line to its graph at the point* $(x_0, f(x_0))$. Vinner (1983) observes that the notion of *tangency* is often strongly linked to problems about the construction of tangent to circles. The approach to those problems focuses on global geometric relationship of the curve and the line, particularly, on the number of points of intersections. Thus, the idea of being tangent—to “touch” in one single point—is featured in opposition to the idea of being secant—to “cut” in two points. This may lead to a narrowing of the concept image of a tangent that is not consistent with the notion of tangent in infinitesimal calculus.

An alternative approach, based on the notion of local straightness, has been proposed by Tall (e.g. Tall, 2000). This is grounded on the fact that the graph of a differentiable function ‘looks straight’ when magnified on a computer screen. Tall claims that local straightness is a primitive human perception, deeply related to the way an individual looks along the graph and apprehends the changes in gradient. However, the notion of local straightness is also a description for the concept of derivative, since it comprises limitations that can trigger theoretical-computational conflicts. As we pointed out in Giraldo et al (2002), these conflicts are related to the fact that finite algorithms are used to describe an infinite limit process. These intrinsic limitations may lead to narrowed concept images, if computational descriptions are over-used. Nevertheless, our hypothesis is that a suitable approach, where theoretical-computational conflicts are not avoided, but highlighted, can prompt the positive conversion of these same limitations: they can make for the enrichment of concept images, by underlining that *the notion of limit, in the sense of infinitesimal calculus, is beyond computers accuracy, no matter how good it is, or, more generally, any finite accuracy.*

In more general terms, we believe that limitations of the various descriptions need not necessarily lead to a narrowing effect. On the contrary, the emphasis on the associated conflicts can stimulate learners to build up richer concept images.

A CASE STUDY

The experiment reported in this section is part of a wider study, in which six first year undergraduate students from a Brazilian university were observed in personal interviews dealing with theoretical-computational conflict situations from different natures (software *Maple V* was used). We summarize the responses of one of the participants, Antônio (pseudonym) to four interviews, concerning the concept of derivative (translated from Portuguese).

Interview 1: Participants were given a few general questions concerning their conceptions about functions, continuity and differentiability.

Antônio was asked how could he decide whether a function is differentiable or not, given the algebraic expression. He stated that it would be differentiable if he could apply known formulae to evaluate derivatives. He was then asked how could he decide about the differentiability if the graph on a computer screen was given, instead of the expression. He stated that he would zoom the graph in to have a more careful view, but it would be impossible to be sure, as computers are not flawless.

Interview 2: Participants were asked to gradually zoom in the graph of the function $y = x^2$ around the point (1,1) and explain what they were observing. For small graphic windows ranges, the graph would acquire the aspect of a polygon, rather than a straight line, as expected (see Giraldo et al, 2002).

Antônio declared he would see something similar to the tangent straight line. When the software started to display a polygonal for the curve, he claimed that the computer was wrong. After thinking for a while, he explained the computer's error:

Antônio: It's because the computer hasn't got idea what it's doing. It's kind of messing up the points. ... As the computer sketches the graph by linking the points and these points are results of approximations, so it links without thinking. It links the points, and whatever it gets will be the graph for it.

Interview 3: Participants were asked to zoom in the graph of the blancmange function around a fixed point using the software *Maple*, and explain what they were observing. The blancmange function (figure 1) is defined in the interval [0,1] as the sum of an infinite series of modulus functions and is continuous but nowhere differentiable (see e.g. Tall, 1982). However, a finite truncation of the series was being used to draw the graph so that the function displayed was non differentiable at a finite set of points, rather than everywhere. The students were familiar with the

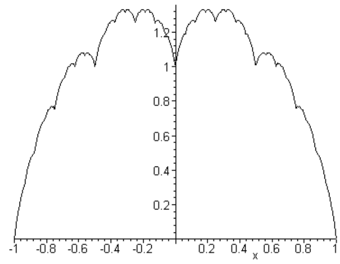

Figure 1: The blancmange.

functions and its properties, as they had studied it previously on calculus lessons.

Antônio started by explaining the construction of the blancmange function. He showed good comprehension of the process:

Antônio: You are taking a number and multiplying it by $\frac{1}{2}$, taking that one and multiplying by $\frac{1}{2}$, by $\frac{1}{2}$. So, it's a geometric progression with rate $\frac{1}{2}$. ... Then, it's the sum of a geometric progression. The sum of a geometric progression is a limit, then it converges to a point. ... Then each point there is a geometric progression, it's the limit of a convergent geometric progression. It's there. ... It's well defined.

He then started the process of local magnification and explained that, as the curve was not differentiable, the graph would become more wrinkled as he zoomed in. As the algorithm used a finite truncation of the series, it did not looked more wrinkled, as he expected, but quickly acquired a straight aspect. Antônio showed great surprise, and asked the reason for the unexpected result. After listening to our explanation, he commented, with increasing excitement:

Antônio: Oh, I see. You could sum a few more steps, but not until infinity. … But it [the computer] can't make infinity. ... Hey! I think that nothing could make! ... It can't add until the infinite! There will be always an infinity missing. And nothing can represent the infinity, as a whole, but we can show that it goes to that place, that it tends to that. That's the infinite. ... It's impossible to represent it, not on the computer, or on a sheet of paper, or anything else! The computer only represents things that a human being knows.

Interview 4: Participants were asked to investigate the differentiability of the functions:

$$v_1(x) = \begin{cases} x\sin(1/x), & if\ \ x \neq 0 \\ 0, & if\ \ x = 0 \end{cases} \quad \text{and} \quad v_2(x) = \begin{cases} x^2\sin(1/x), & if\ \ x \neq 0 \\ 0, & if\ \ x = 0 \end{cases}$$

For that purpose, they were given the graphs of the curves $y = x\sin(1/x)$ and $y = x^2\sin(1/x)$ sketched by Maple in a neighbourhood of the point (0,0) (figure 2).

Figure 2: The curves $y = x\sin(1/x)$ **and** $y = x^2\sin(1/x)$**.**

Antônio said at first that both the functions should be differentiable, as the formulae he knew applied to the algebraic expression. He then started to zoom in the first graph around the origin, and the curve progressively looked more smudged. Antônio argued that again it should be due to an interpolation error, but the function v_1 should have a derivative. Afterwards, he repeated the process for the second graph. He commented:

Antônio: Look, when it gets closer to 0 it kind of tends to an area. But it's not. We can't see it, but it's the joining of two curves with ... the oscillation tends to zero, that's why we cannot distinguish.

We asked Antônio to conclude about the differentiability. He said:

Antônio: If it were $\sin(1/x)$, without anything else, they wouldn't be. They wouldn't be differentiable at 0, because $\sin(1/x)$ wouldn't be defined. But, for these functions the point (0,0) exists, so it's the joining of two curves there. ... Hey, wait a minute! I think v_1 is not [differentiable], do you know why? Because at 0, it's shaped by the joining of the two straight lines, $y = x$ and $y = -x$. ... When it gets closer to that point the parts approach each other within those lines! They will meet each other at that point, right? But it's a rough joining, it's kind of a corner. ... The other one [v_2] is different, it's a smooth joining. Here, the parabolas shape the curve, not the lines, that's the difference. For that reason, I think that one has a derivative and the other hasn't, v_2 has and v_1 has not. ... But I can't be doubtless sure just looking at the graph. Let me think.

Antônio concludes that the only way to be sure would be using the definition of derivative. He has a little difficulty in evaluating the limits, but reassures himself that it would be the only safe way, even if he could not do it.

DISCUSSION

Antônio clearly expressed his preference for algebraic descriptions. He states that the criteria for deciding about the differentiability of a function must be based on formulae. Moreover, he appears to be aware of computers' limitations. Such mental attitude gave him means to quickly grasp the cause of the unexpected result on interview 2. Thus, the involved conflict was almost immediately solved. On the other hand, in interview 3 a conflict played a central role on his reasoning. In fact, his enthusiasm suggests the conflict actually triggered a new idea: *it is not possible to represent infinity by any physical means*. Moreover, he points out the reason for the impossibility: *infinity can never be attained*. The conflict led Antônio to grasp not only the limitations of the computational description, but of other forms as well; and to figure out a conceptual distinction between finite and infinite. The conflict involved in interview 4 was slightly more intricate. In addition to that, the differentiability of the function could not be established by a careless use of the algebraic formulae, against Antônio's dominant criteria. However, the confrontation of computational and algebraic descriptions—suggesting different conclusions—impelled him to follow another strategy: he states that *the differentiability of the function could only be doubtless concluded by means of the formal definition*.

Antônio's mental attitude contributed to the results reported in this paper, suggesting that the conflict have acted as positive factor for the enrichment of Antônio's concept image. Other participants show quite different behaviours. In some cases, the conflicts do prompt students to engage into a rich reasoning. In others, they are barely noticed, as they are quickly solved (like Antônio did on interview 2). But some students very often cannot cope with theoretical-computational conflict situations at all. This obstacle may be due to a more general attitude towards technological devices. The global results of the investigation in which this experiment is comprised are currently being analyzed. One of our aims is to understand more clearly in which situations conflicts do have a positive role and, in particular, in which sense and in which extent learners' previous attitudes and background determine that role.

The main goal of this work is to put forward an alternative model of approach, not purely grounded on formalism nor on imprecise representation forms. It does not mean to undervalue of the formalism, in relation to the imprecise. On the contrary, through the emphasis of limitations and differences, we intend to prompt the development of rich concept images, as well to stress the central role of the formal conceptualization on the construction of a mathematical theory.

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