EXTENDING A SEQUENCE OF SHAPES: PICTURES, PATTERNS AND PROBLEMS

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We consider children's responses to a sequence question from the 2001 Key stage 2 National Curriculum tests. The most common method of successful solution involved some form of table of numbers. Other methods included drawing and use of a relationship. The idea of a 'best method' proved problematic, as both the apparently sophisticated and reliable methods produced errors.

INTRODUCTION

This paper concerns the responses of 11-year-olds to a written question from a National Curriculum test concerning a sequence of growing shapes, which we call 'Squares and Circles' (see Appendix 1). The work arises from a wider study, carried out with the Mathematics Test Development Team at the Qualifications and Curriculum Authority. The study concerns the responses of 11-year-olds to tasks that can be seen as pre-algebraic. The first phase is based on the responses of children to Key Stage 2 written mathematics tests. The selection of questions and responses to some other questions are described elsewhere (Houssart and Evens 2002).

BACKGROUND

Sequences of patterns are seen by many as a way of approaching algebra (eg. Mason 1985, 1996, Lee 1996). Orton et al (1999) discuss the possible benefits of setting pattern tasks within pictorial and practical contexts. These include adding meaning to the task as well as perhaps making it simpler for some or all pupils. Mason et al (1985) make extensive use of patterns of shapes when suggesting activities which will encourage pupils to express generality. They suggest four stages in this process: seeing a pattern; saying a pattern; recording a pattern and testing formulations.

Tasks of this type are also seen by many as appropriate for both primary and secondary children and hence several studies compare the response of upper primary and lower secondary pupils to items involving sequences of patterns. For example, in tests set in 1982, the Assessment of Performance Unit asked 11 year olds and 15 year olds several questions involving sequences of shapes (APU undated). In all questions more 15 year olds than 11 year olds were successful. Pupils were less likely to be successful as the information asked for became further from the pictured shapes. The omission rate was low for questions requiring a number as an answer, but higher when explanations and generalisations were sought.

Stacey (1989) reports a study in which students aged between 9 and 13 worked on what she called 'Linear Generalising Problems'. She classifies pupils' methods and models, including those leading to incorrect answers. These include the 'whole object method' where children take a multiple of the number of parts in a smaller shape. In a more recent study by Orton (1997), children aged 9 to 13 worked with sequences of matchstick shapes. She concludes that there are many barriers to generalisation.

In studying solutions to problems where students have to identify the 100th pattern in a sequence, Ishida points out that drawing is a poor strategy in such problems. He explicitly identifies the 'best method' which is providing an expression linked to a simple generalisable structure (Ishida 1998, in Japanese, reported in Ishida 2002).

The question being considered here differs from those asked in the studies described above in three ways. Firstly the question is presented with a table of numbers alongside the pictures, which may have encouraged the children to use tables or lists of numbers in their solutions. Secondly our question only requires an answer about one other shape in the sequence and it is near enough to be reasonably reached by a drawing or difference method. Stacey calls this a 'near generalisation' and both her study and that of Orton include 'far generalisations' where such methods are unlikely to be practicable. Finally the 'Squares and Circles' question requires pupils to give the number of squares (effectively the same as the number in the sequence) for a given number of circles, rather than the other way round. In this respect the question can be seen as more demanding.

FINDINGS

Overview

We looked at the responses of 451 children to this question. These are summarised in Tables 1-3 in Appendix 2.

This was amongst the harder questions in the test, with only 37% of the papers examined showing the correct answer, as shown in table 1. However, unlike some other 'hard' questions, many children did attempt to answer, with 48% of papers examined showing an incorrect answer. We classified the correct answers according to the method used. We also looked at incorrect answers and the accompanying working, if there was any, to try to find explanations for children's difficulties.

Correct Answers

In this question, children were specifically instructed to show their working in a box provided for this purpose and the majority of them did so. This led to rich data, with diagrams, words, numbers and occasionally symbols used by way of explanation. Initial analysis suggests a wide range of solutions. A summary of solutions used by those giving the correct answer is shown in Table 2. The first, and apparently simplest category of working we call 'Diagrams'. Most children giving answers in this category drew the pattern using 25 circles and then counted the squares. Other common solutions involved some type of table, chart or list of numbers. Some children seemed to have worked downwards, continuing both columns until they arrived at 25 circles. Others showed evidence of working across, linking the number of circles to the number of squares in each case. Such evidence was in the linking of the pairs of numbers by lines or rings, or the use of ordered pairs. Almost half the children who answered this question correctly used some sort of continuation of number pattern in a table. Finally, some children presented solutions that focussed on the relationship between the number of circles and the number of squares. These included explanations about subtracting one and dividing by two and in a few cases made use of letters to express the relationship.

Incorrect Answers

Analysis of incorrect answers is shown in Table 3. As with other questions we looked at, there was a wide range of incorrect answers, sometimes without working, many of which may have been guesses. However this question did produce some relatively common incorrect answers, some of which included working or explanations. They suggest four common errors. The most common of these was to assume one square to every three circles, arriving at an answer of 8, 9 or something in between. 47 children gave answers in this range, including 10 who made use of diagrams. A more surprising common incorrect answer was 10. This is explained in one of the examples in Appendix 1. This can be seen as similar to the answers of 8-9, as it was based on one diagram from the sequence only, known by Stacey (1989) as the ‘whole object’ method. A less common incorrect answer was 13, perhaps arising from children trying to halve 25. Finally, some children gave the answer 51, based on finding the relationship between the number of circles and squares, but applying it the wrong way round.

DISCUSSION

Children’s responses to this question differ to similar questions reported in the research literature in that the most common successful strategy was to use some sort of table of numbers. However it could be argued that children were drawn to this strategy by the fact that a table was effectively started for them. The fact that this question involved a ‘near generalisation’ meant that many children solved it by drawing and some also made use of the relationship between the number of circles and squares. The most common errors involved the ‘whole object’ method, which is consistent with other research. An additional error, caused by the fact that this was an ‘inverse’ problem, was to apply the relationship the wrong way.

It is difficult to define a ‘best strategy’ for this problem. Using the relationship between the numbers of circles and squares can be seen as most sophisticated method and would certainly be preferable in the case of a ‘far generalisation’. However in this case, some children applying the ‘relationship’ approach did so the wrong way round and arrived at an incorrect answer.

The drawing approach could be seen as the least sophisticated, but most reliable. However for some children, there is a suggestion that drawing the shapes may have aided their understanding of the relationship. This method was far from foolproof, with some children drawing an incorrect pattern.

As well as accuracy, strategies can also be considered in terms of whether they helped children see the structure of the pattern, though this is not something easy to determine from a written answer. There is a suggestion that some children making correct use of drawings became more aware of the structure of the pattern as they drew. This is evident in the increasing gaps between shapes and in the addition of a relationship in one case.

Finally it is important to remember that we can not be certain that the method presented in the solution box is the one that the child actually used to reach the solution. Some children presented a formal but incorrect method, such as dividing by three, alongside the correct answer.

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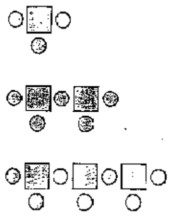
APPENDIX 1

The Question (right)

Some solutions giving the correct answer (below left)

Common incorrect answers (below right)

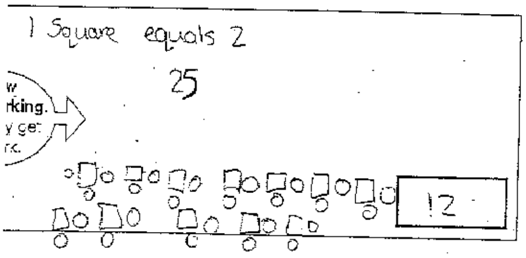
23 Here is a sequence of patterns made from squares and circles.

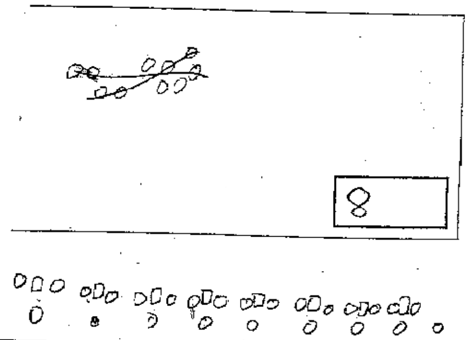


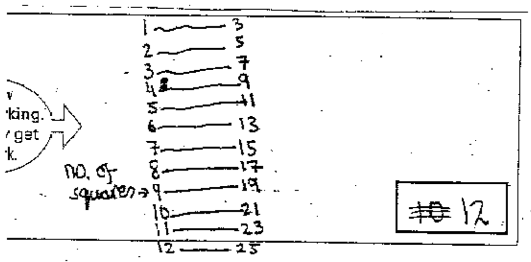
number of squares	number of circles
1	3
2	5
3	7

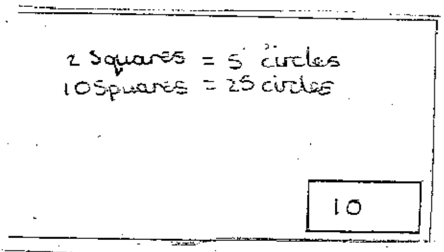
The sequence continues in the same way.

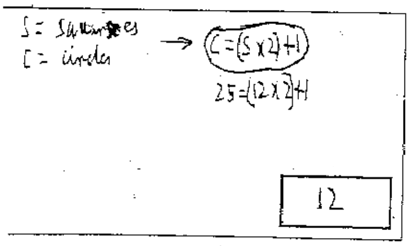
Calculate how many **squares** there will be in the pattern which has **25 circles.**

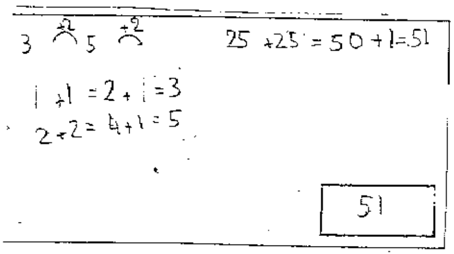












APPENDIX 2 : RESULTS TABLES

2001 KS2 Paper A Question 23 Squares and Circles

Table 1

Total number of scripts 451		
	Number	Percentage (to 1%) of total number of scripts
Correct answer	168	37
No response	65	14
Incorrect answer	218	48

Table 2

Analysis of correct answers (168 scripts)		
Description	Number	Percentage of correct answers
Correct 12 No working	15	9
Diagrams	38	23
Using differences between no. of circles and no. of squares i.e. adding 1 more each time	3	2
Adding 2 to the number of circles but no record of number of squares.	10	6
Table containing number of squares and number of circles but no linking.	34	20
Evidence of pairing the number of squares with corresponding number of circles e.g. ordered pairs, rings or lines joining.	33	20
Evidence of relationship × 2 +1	9	5
Evidence of – 1 ÷ 2	5	3
Other working	21	13

Table 3

Analysis of incorrect answers (218 scripts)		
Description	Number	Percentage of incorrect answers
Answer 13	10	5
Answer 51	9	4
Answer 8, 9 or similar	47	22
Answer 10	30	14
Other answers	122	56