INVESTIGATING STUDENTS' UNDERSTANDING OF LOCUS WITH DYNAMIC GEOMETRY

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Dynamic Geometry (DG) offers new possibilities for the investigation of loci. The two functions "Locus" and "Trace" provide complementary tools for thinking about loci: "global" in the Euclidean sense, "local" in the pointwise sense of analytic geometry. By using Trace the concept of functional relationship can be explored, an idea which is neglected in the current curriculum. We report on a small experiment with 15 year old students, in which we compared the students' understandings of the 'local' and 'global' structure of loci under the different cognitive and cultural influences of working with conventional tools (compass and straight-edge) or DG.

INTRODUCTION

This research is a preliminary study for a PhD (by Shinwha Cha) in the area of using DG to look at simple locus problems and conic sections. The overall aim is to link Euclidean geometry and analytic geometry, which tend to be regarded by students as separated subjects, often poorly understood. But, by connecting them for the students, the hope is that they will understand both better.

This study was carried out to answer two main questions: What kinds of existing knowledge do students have about locus? What are the new aspects that emerge within a DG environment of students thinking about locus?

Formally, a locus is a path traced by a point as it moves so as to satisfy certain conditions. The examples first encountered by students are typically the circle, perpendicular bisector, and angle bisector. It seems that students are often simply taught to solve these simple locus problems in terms of a geometric construction, neglecting the 'point-wise' structure. Students develop a superficial impression about locus: 'To solve this locus problem, construct this geometrical figure'. For example, to find the locus of points equidistant from two given points, you construct the perpendicular bisector. Or, as one student put it, "You just find two points and then join them up." This is a 'global' understanding of locus, seeing it as a whole shape, in contrast to a 'local' understanding, which means seeing the properties of individual points on the locus. The pointwise idea is fundamental in analytic geometry, beginning with an arbitrary point which satisfies given conditions and then generalising the point into an algebraic form so that the locus can be plotted in a Cartesian system.

It is critically important for students to see and translate the idea of locus in analytic and synthetic ways across different representations. This aspect of the study reflects research that has pointed out how learning mathematics generally means understanding mathematical concepts across various representations (Duval, 1999; Arcavi & Hadas, 2000).

EXPERIMENTS

This experiment was carried out with two 15 year old students in a comprehensive girls' school in London. They were provided with worksheets that consisted of two tasks to be attempted, first using compass/straight edge and then with Cabri:

Task 1: Find the locus of points which have the same distance from two given points.

This task is familiar from the classroom and they have good intuitions about it.

Task 2: ("Apollonius circle") Find the locus of points which have twice the distance from one given point as the distance from a second given point.

This task is new and difficult for their intuitions. Although these two tasks are very similar, the solutions and the strategies to solve them are quite different, unless you introduce a coordinate system and use algebra. There is no single straight-forward geometric method for solving both problems.

Episode 1: Using dragging to give a different feeling about locus

This episode is taken from an activity with Task 1 in a Cabri environment. The students had already learnt how to construct the perpendicular bisector in the classroom, and they had seen it as a locus question in national tests, although they did not know how it worked. We felt they had a weak understanding of the "pointwise" idea of locus, so this activity was designed to give them a different feeling about locus in which the pointwise approach is emphasized through the Cabri action of dragging. Also, students have a very strong habit of measuring distances, so we made measuring explicit in the construction, so that they could build on what they already know.

The 'pointwise' approach is made explicit here (Figure 1) without using the commands Locus or Trace, which are difficult for students who are encountering Cabri for the first time.

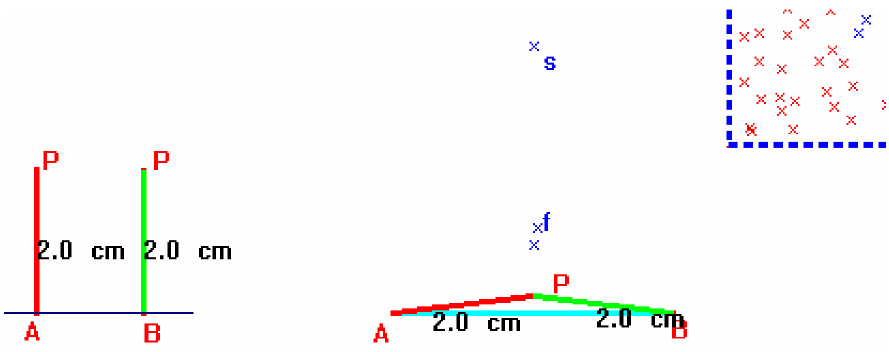

Figure 1

The point P can be dragged by the mouse, and the two bars on the left represent the lengths of the segments AP and BP, dependent on the point P. When the student finds a point P that has AP = BP she marks it with a cross (×), dragged from the box on the right. (Arzarello et al, 1998, describe this kind of systematic dragging as "dummy locus", in which "dragging can act as a mediator between figures and concepts", ibid, p. 37).

The first point the students found was the one marked "f" (figure above). They placed further points, but only on the one side of the segment AB. There was some evidence of pointwise thinking here from the students' written response: "AP=BP will be the case anywhere above or below the main line AB. For the line to be equal, they need to be directly in line with the centre of AB".

Episode 2: The Apollonius circle with dragging

This is the same situation as in Episode 1, except that the ratio AP:BP is 2:1. It was made easier for the students because they needed to look for equal bars when the lengths are 2:1 (Figure 2). The points marked in below indicate the order in which the students found them. Notice that they did not first find the 'internal' and 'external' points on AB, which one would normally start with in an analytic approach.

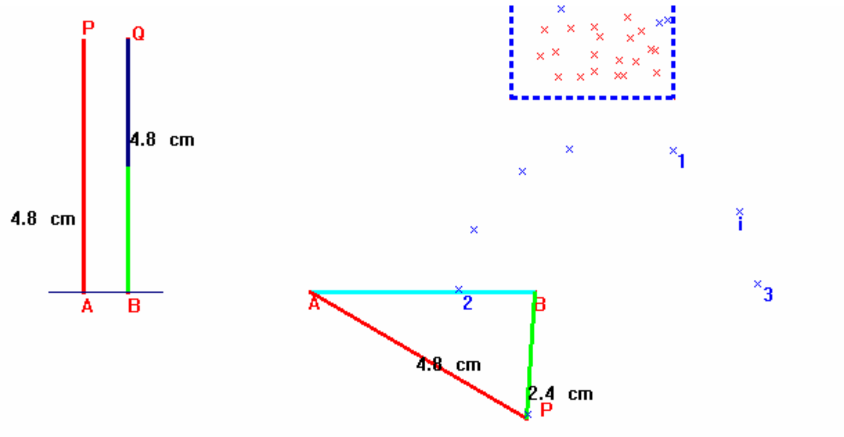

Figure 2

At this point, the students' written response was clear: "It has been shown that the set of points form a circle". However, in the next episode we will see how a different Cabri situation provoked a different and more dynamic response.

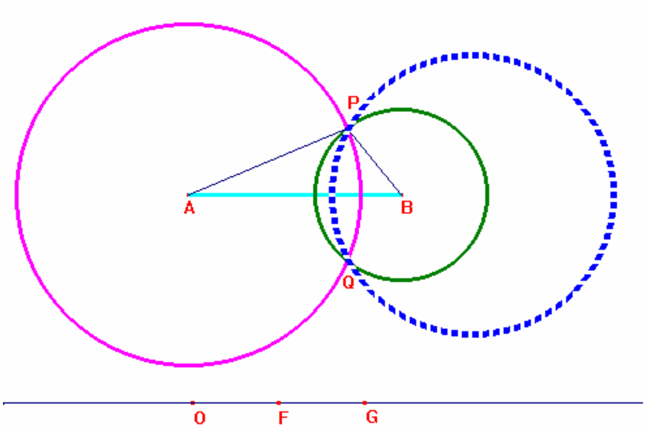

Figure 3

Episode 3: The Apollonius circle with Trace

We introduced the Trace command in Cabri rather than Locus, because we thought Locus would be too difficult for these students to use with their limited experience of Cabri. Given the setup as shown in Figure 3, the students were firstly asked to make a guess about the shape of the locus as the control point F is dragged along OG (which determines the radii AP and BP). Before turning on Trace, their first conjecture was that the locus might be a circle or an arc. But, suddenly, they shifted to say it is a line.

Alice: It's like an arc isn't it?

Jackie: It would be a circle. Can you move it [F] further?

Alice: (*repeatedly*) Always arc.

Jackie: They get, close it together. It's just part of a circle. It always is a circle.

Alice: It's a straight line. It's straight line isn't it? It is between P and Q. *(keeps saying 'straight line'*).

Jackie: Agreed.

They turned on Trace and then they saw the circle. They said, "Uuh no its a circle". When they were asked why it's a circle, they replied as follows:

"The distance between P and Q is a straight line. But they [the circles] move, they are together and then they are separate [and then] together again"

Clearly, they are using some mental image of a line. Of course, they have a limited range of experience to explain and make sense of geometric images. For example, in their experience having two points is often a cue to join them up together, and this becomes a prototype of a tendency about what they are supposed to do with two points. Also, they had worked on Task 1 (the perpendicular bisector) a few minutes previously in which they had joined two intersection points P and Q to make a line. However, notice that they say

… they are together and then they are separate [and then] together …

They are *imagining* the line joining up points P and Q as they drag the point F, and have described the locus in these terms, see Figure 4.

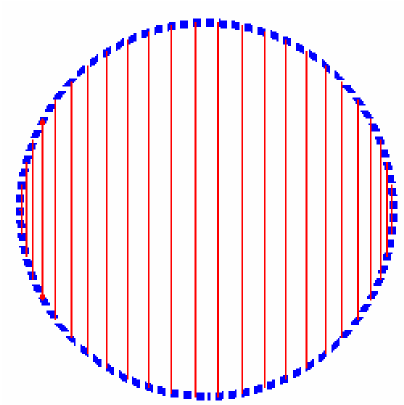

Figure 4

They seem to feel easier to see the line in their minds, and in a follow-up interview they still said that they feel more comfortable to understand the locus in terms of the change of a line segment moving across the circle.

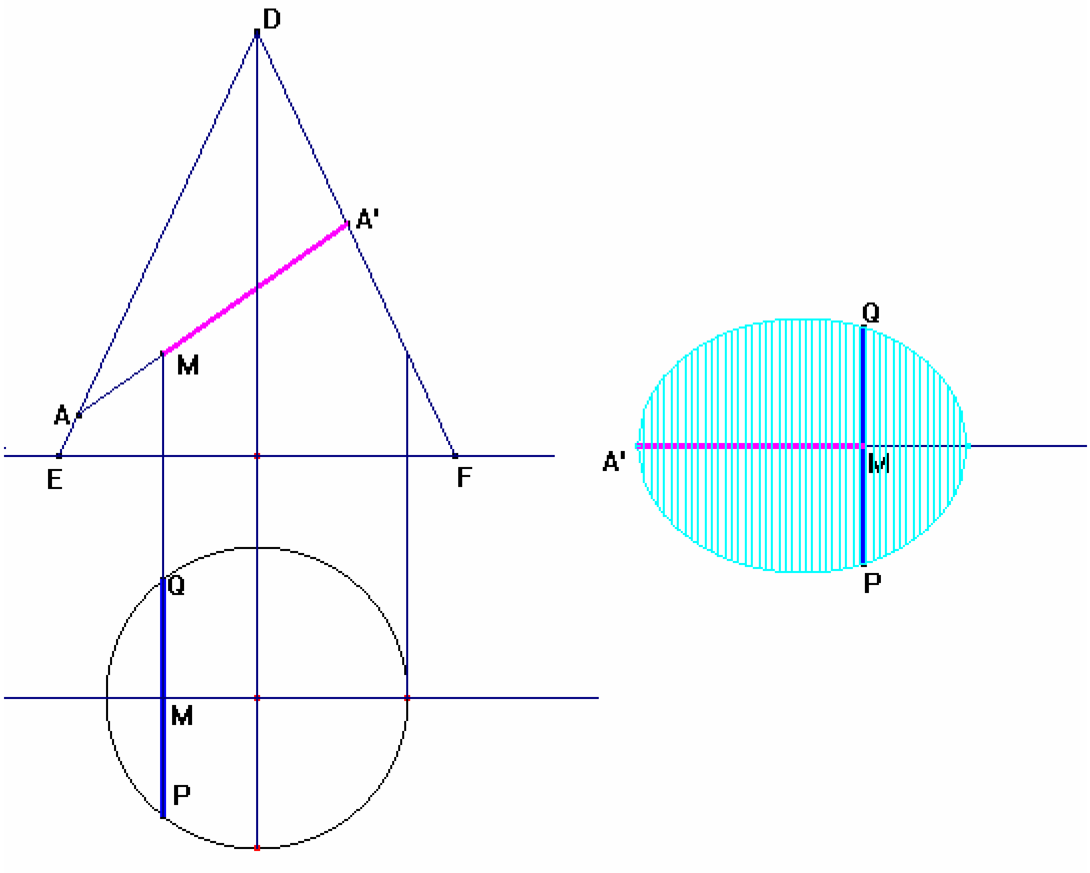

Figure 5

DISCUSSION

We think that the students really have discovered a different way of making sense of the Apollonius circle in this DG environment. Seeing the dynamic image of the points P and Q moving seems to have stimulated their geometric intuition in a novel way. Although it is not 'pointwise', it is a 'local' understanding. This kind of intuition, something like 'slicing a disc', can be found in the history of mathematics. It is very like the techniques of Apollonius and Durer for constructing conic sections, which use a pre-Cartesian form of coordinates. Kepler in the 17th century used the idea of 'slicing' to find a formula for the area of an ellipse, which was a significant step in the development of the calculus (Boyer, 1968).

Durer (1525) described a method to construct the conic sections of a right circular cone (Pedoe, 1976). His method is very similar to what Apollonius used in his explanation of conic sections (Boyer, 1968, pp. 164-5). Durer's approach seems relevant to what the students did, because it is a rather intuitive technique to draw a 'graph' without using algebraic representation. Figure 5 shows what is happening with Cabri. Suppose you slice the cone with a plane. The triangle DEF shows the situation at the centre of the cone, where the plane cuts the cone at points A and A'.

We make an object point M on the segment AA'. The distance A'M is the first quantity that we want to plot. For each value of A'M we need to know the width of the conic section at that point. Durer finds this by drawing the circle corresponding to the height of the cone at that point. Then, he finds the width PQ by locating the intersection of a horizontal line through M with the circle. Finally, on the "axis" A'M we add the segment PQ perpendicular to A'M. And if we do Locus, we can see the locus of conic section.

We think it can be argued that the student's intuitions resonate in some way with the historical stage of geometry before Descartes. As Boyer notes,

> there appear to be no cases in ancient geometry in which a coordinate frame of reference was laid down a priori for purposes of graphical representation of an equation or relationship, whether symbolically or rhetorically expressed. Of Greek geometry we may say that equations are determined by curves, but not that curves were defined by equations. Coordinates, variables, and equations were subsidiary notions from a specific geometrical situation. (Boyer 1968, p.173)

The students seem to have an intuition for 'Apollonian coordinates'. We are currently investigating in what ways using Apollonian coordinates could be an intermediate step towards Cartesian geometry, working in DG. That is, whether there is a way in which students can make a genetic approach to graphs and functions, in which students investigate the coordination of quantities by geometric construction in a DG environment, before using algebraic symbol representations.

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Note: The Cabri files used in the study can be downloaded from: http://website.lineone.net/~shinwha.

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