SYMBOL SENSE:
TEACHER'S AND STUDENT'S UNDERSTANDING

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During 1999-2000 research into student's symbol sense following Arcavi's definition (1994) and using common errors and misconceptions as a basis for a questionnaire was undertaken with high attaining female students aged 15 to 18. The same questionnaire has been used with experienced teachers and trainee secondary teachers.
The findings have some important implications for the teaching of algebra that might help to develop symbol sense in both students and teachers - reading sense into symbols and understanding the nature of symbols, functions and variables and relationships between representations.

INTRODUCTION

Arcavi (1994: 32) introduces the notion of symbol sense as a 'desired goal for mathematics education'. Symbol sense incorporates the ability to appreciate the power of symbols, to know when the use of symbols is appropriate and an ability to manipulate and make sense of symbols in a range of contexts. The premise of this research is that a person with symbol sense is less likely to make common errors and exhibit common misconceptions when working with algebra. By exploring the responses of teachers and students to items based on common errors and misconceptions it is hoped to establish the extent to which people in these two groups have symbol sense. The outcomes of the research may also offer insights into specific areas of algebraic understanding where more effective pedagogic approaches need to be developed.

METHODOLOGY

The sample of students comprised thirty two high attainers from an Independent Girl's school in an affluent suburb of London, it was composed of eight pupils aged 15, 16, 17 and 18 years. The sample of teachers and student teachers was just twelve, three were experienced and nine trainee secondary teachers.
The instrument was a ten question questionnaire, conducted in January 2000 with the students and in Spring 2001 with the teachers. The questionnaire is based on classic errors and misconceptions from research and examiner's reports. The responses were allocated scores and analysed using a spreadsheet. Statistical testing with such a small sample was not viable.

THE STUDY

In this paper we discuss just five questions from the questionnaire to illustrate the differences and similarities in the responses between teachers and students.

Question 1

There is considerable evidence in examination reports e.g. QCA 1997, QCA 1998, QCA 1999 and Edexcel 1998 that students at ages 14 and 16 have difficulty understanding and using indices in algebraic expressions. The following question was designed to explore this.

What does $2y^2$ mean?

Tick all the answers that you think are right

$2\times y\times 2$	$4\times y\times y$	$\left(8^{\frac{1}{2}}y^3\right)^{\frac{2}{3}}$
$2\times 2\times y\times y$	$\sqrt{4y^4}$	

All students and teachers recognised $2\times y\times y$ as equivalent to $2y^2$.

Just 23 of the 32 students recognised $\sqrt{4y^4}$ as equivalent to $2y^2$, whereas all 12 teachers correctly recognised the equivalence. However only nine of the 32 students recognised $\left(8^{\frac{1}{2}}y^3\right)^{\frac{2}{3}}$ as equivalent to $2y^2$, and only 5 of the 12 teachers did so.

The results suggest that as students and teachers gain in experience they are more likely to be able to deal with indices correctly, however, the fact that the majority did not recognise this equivalence suggests that using and understanding fractional indices and powers of powers is an area worthy of further research.

Question 2

The concept of a symbol as a variable 'requires an ability to think simultaneously about entire families of numbers rather than about a specific quantity' (Sfard, 1995: 21). The work of English and Warren (1998) also suggests that many expressions are difficult for students to understand if they involve operations other than one stage addition or subtraction.

a) y^2 b) $\sqrt{y}$ c) $\frac{1}{y}$ d) $y+1$ e) $\frac{y}{0.5}$ f) $y-0.5$

Which of these expressions are **always** greater than *y*?
Which of these expressions are **sometimes** greater than *y*?
Which of these expressions are **never** greater than *y*?

The majority of students (all except two 16 year olds) and all of the teachers correctly identified $y+1$ as **always** greater than *y*. One quarter (3 out of 12) of the teachers incorrectly identified $\frac{y}{0.5}$ as always greater than *y*, compared with half of the students. There is evidence to suggest that this error declined with the age of the students as just two of the 18 year old students made this error compared with six of the 16 year olds.
Sometimes greater than *y* caused the greatest problems to both students and teachers, particularly with the square root and the reciprocal functions.
For **never** greater than *y* all the teachers and older students correctly identified $y-0.5$. Five of the younger students failed to identify this expression as never greater than *y*, although those who were interviewed were able to correct their error. Again errors arose with the reciprocal and square root functions with two of the trainee teachers and one teacher incorrectly identifying these and fifteen students identifying the square root function and eight students identifying the reciprocal function.
Some of the errors made in response to this item could be attributed to the type of variable that *y* is supposed to be. Certainly the interviews with students suggested that a common misconception was to assume that *y* was an integer or even a natural number.

Question 3

There are 9 times as many pupils in a school as there are teachers. *P* represents the number of pupils and *T* represents the number of teachers in the school.
Put a tick next to any of the statements below that are true

$P=9=T$	$9T=P$	$\frac{T}{9}=P$
$9P=T$	$\frac{P}{9}=T$	
$T+9=P$		$P^9=T$

This item was a word variation of the well documented ‘students and professors problem’ and is used by Arcavi (1994: 27) as an ‘example of reading for reasonableness’. Two of the statements transliterated the verbal statement into algebra. Arcavi would argue that anyone with symbol sense would not select these options!
The majority of students and teachers correctly selected $9T=P$ and $\frac{P}{9}=T$, with the success rate improving as the age of the students increased. However three of the trainee teachers made the classic error of selecting

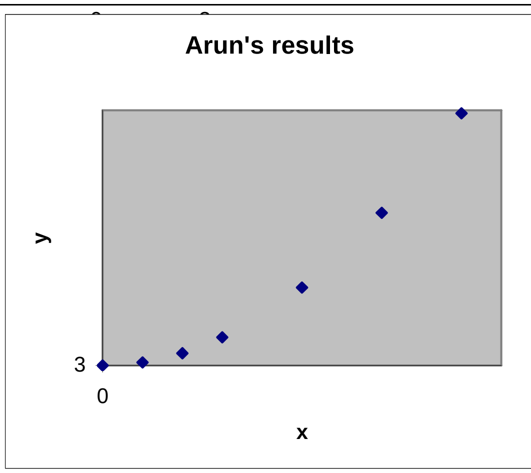
$9P = T$. Only two of the other expressions were selected: $\frac{T}{9} = P$ and $9P = T$, both of which respresent the error of transliteration. The results suggest that with increasing maturity and experience of both students and teachers there is less likelihood of transliteration.

Question 4

I think of a number, add five, multiply the answer by four, subtract twenty and finally divide by four. The answer is the number I started with. Can you prove the answer is always the number I started with?

This question was based on Hoyes and Healey's work on proof (1999) and was used to see if appropriate use of a symbol to represent a number could be used to present a rigorous mathematical argument as to why the trick works.
Six of the thirty two students did not attempt to incorporate any deductive reasoning into their attempt at proof. The most common error in partially correct proofs was to assume equality of their algebraic expression with the unknown number. This was made by eight students and four teachers (two of whom were trainees). This suggests that the role of the equals sign is not well understood. This has been the subject of considerable research, some of which is discussed in Nickson (2000). Half the students (again with older students being more successful than younger students) and two thirds of the teachers successfully completed a proof making accurate and correct use of symbols throughout.

Question 5



Arun performed an experiment in science and wanted to find an equation relating x and y. He obtained some results and plotted them on a graph. Here is his graph. Which of the following do you think could represent the equation relating x and y?

$y = 5x + 3, y = x^2 + 3, y = 3x^2, y = 4x^2 + 3,$
$y = (x + 3)^2, y = x^3 + 3, y = 2x^3 + 3,$
$y = \frac{1}{x} + 3$

Arcavi (1994:31) argues that symbol sense includes the ability to 'successfully engineer symbolic relationships which express the verbal or graphical information needed to make progress in a problem'. The APU study (1980) found that only 22% of 16 year olds could select the graph to represent $y = (x-1)(x+4)$.

The results revealed a surprising lack of visual awareness amongst both students and teachers. Five of the younger students and three of the trainee teachers thought that the linear expression was a possibility! This suggests that the connection between a linear expression and its graphical representation as a straight line is not well embedded. However, when interviewed one of the students did recognise that a linear expression was inappropriate for Arun's experimental data.

The majority were able to recognise a one stage translation of $y = x^2$ to $y = x^2 + 3$. However less than half the students and only half the teachers (including trainees) were able to recognise the two stage transformation: stretch and translation to $y = 4x^2 + 3$.

The teachers had less success than the trainee teachers overall and this may be because they have less experience of graph plotters and graphic calculators. None of the younger students and only a few of the trainee teachers thought that cubic expressions were appropriate.

The results certainly suggest that the conceptual difficulties associated with moving flexibly between representations do reduce as students get older and have more experience. One of the 18 year old students suggested an alternative equation: $y = 3e^x$, whilst another couple of students correctly suggested different transformations of cubics and quadratics.

CONCLUSION

The results discussed above suggest that whilst symbol sense, the ability to use symbols with understanding, appears to increase with maturity and experience neither students nor teachers have symbol sense per se. In particular the first two questions suggest that fractional indices, roots and reciprocals when applied as functions to variables are poorly understood. This is in line with the findings of English and Warren (1998) and suggests that greater experience with functions of these types may assist in developing confidence. The results of question 2 also suggest that experience of different types of variables is important – the variable doesn't have to be a whole number. It may be that early experiences with function machines where whole numbers are used as input could contribute to a limited view of what constitutes a variable. This is an area for further research.

The transliteration of verbal statements into algebra (question 3) was a classic error made by both trainee teachers and students in this study. Clearly developing the skills of reading symbols for meaning is something that needs requires careful attention when teaching.

The use of symbols to complete proof (question 4) is another area that requires attention, particularly the use of the equals sign. The incorrect use of equals i.e. assuming equality when attempting proof was made by a small but significant number of both students and teachers. Perhaps the sloppy use of equals in arithmetic (e.g. $2 \times 5 = 10 + 4 = 14$ etc. contributes to this type of misunderstanding.

The ability to move flexibly between graphical and symbolic relationships was found to improve with age, however more exploration using graph plotters and/or graphic calculators may well help to improve understanding of graphical representations of algebraic expressions.

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