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MATHEMATICS AS A CONSTRUCTIVE ENTERPRISE

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Seeing mathematics as a constructive enterprise could open the possibility for learners to contact a creative aspect of mathematical thinking. Asking learners to construct mathematical objects seems initially to require great creativity, yet every mathematical task learners attempt has a creative element, however obscured by memorised and trained procedures.

The aim of this session was to report on research undertaken with Anne Watson, by offering task-exercises designed to draw attention to aspects of mathematical thinking and mathematical awareness which are sometimes obscured by standard tasks perceived in standard ways. They were intended to draw attention to how people construct mathematical objects which satisfy constraints. They were also intended to highlight the mathematical theme of 'freedom and constraint' through experiencing a shift in perception concerning different possible dimensions of variation, and within those, of permissible range of change. This applies both to the mathematical content of the task-exercises, and to their structure.

INTRODUCTION

Mathematics can be seen in many ways: as an expressive and manipulable language, as a world of definitions, theorems and proofs *á la* Popper; as a domain of exploration and discovery; as a human construction *á la* Kronecker; as a collection of social practices; as a collection of procedures to answer classes of problems; as exposing the structure of the universe. The list goes on. Undoubtedly it is this very polysemy which enables mathematicians to contribute to widely diverse fields of human enquiry.

I start from a position that Anne Watson and I have been developing, namely that getting learners to construct objects for themselves is both an excellent way of deepening and enriching their appreciation of mathematical terms, ideas, and techniques, and a useful way to locate areas in which learners are less than confident. In the session I proposed some task-exercises which could give access to some of the psychological aspects of example creation. One thing which certainly arose is that constructing objects seems at first a rather unfamiliar thing to do for many of us, but I went on to suggest that in fact, *all* mathematical tasks of the 'to-find' variety can usefully be seen as construction tasks. Because as teachers we have focused learners' attention on procedures as ways of getting answers (very often unique answers), we and they have been led to overlook and downplay the constructive nature of any procedure, namely, as a recipe for constructing an object that satisfies certain constraints specified in the problem. By emphasising the constructive aspect of procedures, I conjecture that we can make it much more common, and much easier, for learners to construct other kinds of mathematical objects and hence to enhance a creative dimension of mathematics and to enrich their learning.

A NOTE ABOUT METHOD

I choose to report our ongoing research consistently with my approach to this kind of research. The principal product of this research is awareness. Awareness is required in order to be sensitive to opportunities to act differently in the future, but awareness cannot be described directly, only indirectly. Consequently another product (which is constantly changing and developing) is a collection of task-exercises which, on the basis of theories and experience, are expected to bring about a shift of attention, affording an opportunity for the education of awareness, leading, perhaps, to enhanced opportunities in the future to alter habits.

THE SESSION

I therefore offered a sequence of tasks based around the construction of objects, inviting participants to record their constructions, and also any observations they had about how those constructions were achieved. I permitted myself to make some observations which seemed pertinent to what people were doing and saying, and what I was experiencing at the time.

The tasks were intended to offer contrasts, with some leading to the use of standard and familiar techniques (with the explicit suggestion that all techniques are recipes for construction, and the implicit suggestion that learner struggles when asked to construct objects might arise from being enculturated into recipes as procedures rather than recipes as succinct and efficient means of construction.

Digitising

> Write down a number between 1 and 2.
>
> Write down a number between 1 and 2 using each of the digits 0 through 9 exactly once.
>
> Write down a number between 1 and 2 using each of the digits 0 through 9 exactly once which is as close to 1.5 as you can make it.

Note: each displayed line was offered as a specific task, so that the tasks in sequence build up the constraints and complexity, and so restrict freedom of choice of object to satisfy all the constraints. One of the opportunities afforded by this style of task, in contrast to offering simply the final part, is to suggest that there are degrees of freedom, with each additional constraint potentially reducing the range of possible answers. In the process, the notion that there are, initially, many possibilities (and thus a sense of freedom) is suggested.

The version chosen for this occasion was

> Write down a number using each of the digits from 0 to 9 exactly once.

Write down a number using each of the digits from 0 to 9 exactly once which is as close to one-half as possible.

As expected in any group, there were several interpretations, including that the second number should be half of the first. Whenever an alternative interpretation arises, an opportunity appears to learn something about the structure of attention, of how it can be confined and trapped in various ways and to varying degrees.

Diophantos

Write down two numbers

Whose sum is 10

And whose difference is 4

Note: again the sequencing is intended to draw attention to the increasing constraints, and hence the freedom of choice available at the start, and how this changes. Learners sometimes experience only the full set of constraints, and find themselves overwhelmed and stymied as to how to begin. Experiencing this format of building up constraint-complexity suggests a general strategy for tackling other tasks where the full complexity is present from the start.

I then kept the 'sum' at 10 but modified the difference: difference of 8, 9, and 12. I was drawing attention to the range of possible values which might not have been present in people's minds on doing the first one. When learners focus on one task at a time, they may not appreciate the dimensions of variation (Runesson 2001) and the permissible ranges of change within those dimensions, which characterise the class of tasks of which this one is intended to be a generic example. Instead, learners tend to focus on the specifics, and to miss the exemplariness of the example. By being exposed to dimensions of variation and range of permissible change, learners gain access to the generality and hence to the method *as* a method, which supports reification (Sfard 1991, 1994) and proceptualisation (Gray & Tall 1994).

When learners are first shown 'a simple version' of a technique or strategy, and only later shown complicated ones, they may sometimes not pay full attention to the technique because they have other ways of getting the answers to the simple versions. Consequently when complex tasks arrive, learners are left stranded. Here the proposal is that, for learners not yet confident with symbols for as-yet-unknowns, it is possible to develop an approach to 'doing' the tasks which, when expressed in general, whether in words or symbols or both, gives rise to a method, which itself can then be generalised to produce a technique, such as algebraic equations.

Several participants indicated by their responses that they stopped work once they had a solution to the first task, because there was no challenge and no need to carry it through. Others dove into a systematic trial-and-improvement starting from the mean of the sum. This too can in fact be used for solving such general equations, though arithmetically rather than algebraically, and certainly not efficiently in general.

Marbles

> If Anne gives one of her marbles to John, they will have the same number; if instead John gives one of his to Anne, she will have twice as many as John. How many marbles have they each?
>
> Make up a problem like this one. Make up a complicated one using these ideas.

Note: some people changed the names; some people changed one or more of the numbers involved (are the two occurrences of 'one' the same or could they be different?), some altered the mathematical structure, even jumping to 'any context in which there are two equations and two unknowns. If the numbers are altered, will a solution still be possible?

This raises again the question of what it is that learners are aware of when meeting a task. We set learners tasks not because we want to know the answers, nor because it is of value to the learner to know the answer, but because we believe it to be of value to the learner to seek a way of getting an answer. If learners see a task as very particular, then they are immersed in particularities, and one wonders what the point is in doing it, for the important thing is to 'learn from the experience' by feeling that you could do another one like it in the future. But what constitutes 'one like it'?

A great deal can be learned about getting learners to reveal what aspects or features they are attending to, and hence, perhaps, what aspects or features they may be overlooking or simply not seeing. A good way to find out what learners are stressing is to ask them to construct 'another one like it'. However, one must be cautious about drawing conclusions about what they are ignoring through analysing what they do not make explicit. 'Absence of evidence is not evidence of absence!'.

Cubic Sketches

> Sketch a cubic (polynomial)
>
> Sketch a cubic which has a local maximum and a local minimum
>
> Sketch a cubic which has a local maximum and a local minimum and which has three real roots
>
> Sketch a cubic polynomial which has a local maximum and a local minimum and which has three real roots, and has an inflection tangent with negative slope

NOW!

> You may have found one sketch that meets several of these four requirements. Go back and construct an example so that at each stage your example *does not* meet the requirements in the next stage. Thus, your new third one cannot have an inflection tangent with negative slope but must have local maximum and minimum; your new first is a cubic without local maximum or minimum.
>
> Pay attention to how you cope with the 'negations'.

Some participants, for whom the notion of an inflection tangent was unfamiliar, found it hard to go beyond stage three. The time-pressured setting of the session, like time-pressured lessons, was not conducive to exploration of possibilities with a view to locating what it might mean. Some could be heard explaining it to others, some of whom used the clarification or information, while others did not. Many participants sketched and rejected (by crossing out) before alighting on a suitable example. Few thought to label which diagram was which of their seven or eight sketches!

FURTHER COMMENTS

Responses from participants confirmed that where, as in *Diophantos*, a technique was known (either try-and-improve by adjustment, or use simultaneous equations), the technique dominated attention. For some it was enough to recognise the technique, with no desire to resolve the particular question, rather as some people find when asked to *write down a number*: there is no basis for selection, no reason to continue with something which has no challenge, no edge, with something which is entirely automatic and understood.

In our development of these ideas, Anne Watson has suggested a metaphor for object construction. Imagine you are looking for a cooking ingredient so you go to the pantry. You may find what you want where you expect it (the chopped tomatoes for example); you may have to search for it. On the other hand, you may not find what you are looking for, but espy something that might do in its place; or you might have a sense that there used to be something at the back of the shelf, so you reach out more in hope than in confidence; or you realise that what you want will not be there, but you might be able to assemble something using bits and pieces of what is available. Every so often you rearrange the contents of the pantry, thereby reviewing what you have, perhaps even discovering forgotten things. Furthermore, when you are shopping, you may notice something that strikes you as having potential, so you purchase it on the off-chance that it will prove useful.

Most of these states of reaching for something on the shelf can be experienced in the tasks offered. The last two states, of rearranging the contents and of seeing that something might have potential, either in some thing you are planning, or in some much more indefinite way in the future as an ingredient, is difficult to reproduce as an experience. But it seems to describe an important way in which we take up new objects as components for future use. Rearrangement occurs when we make connections and links, perhaps seeing something familiar as a special case of something else, or seeing two previously unrelated objects as being related and worthy of being stored near each other. In a fresh context, something about an object catches attention, which affords access again in the future. It would be useful to know more about what it is that 'strikes me as potentially useful', and whether that could yield insight into offering experiences to learners.

There appeared to be considerable resonance with this metaphoric description, but it was also clear that participants were not used to discussing *how* they constructed objects except in relation to the use of algorithms or procedures. It takes effort and practice to be able to attend to socio-psychological aspects of object construction.

I am grateful to participants for leaving their 'objects' behind for my perusal.

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