SYMBOL SENSE - THE GAP BETWEEN GCSE AND A LEVEL

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Abstract: *For my Masters Dissertation I researched pupils' symbol sense as defined by Arcavi (1994). Before I started I expected to find that, as pupils' matured, their symbol sense would improve. What I had not expected was to see such a clear gap differentiating GCSE pupils' symbol sense from those studying A Level. This paper suggests that, although some A Level pupils still struggle to understand algebra in the sixth form, there is a clear divide in pupils' understanding of algebra between GCSE and A Level, a gap which new sixth form pupils have to cross.*

Introduction

Wiliam (1996) suggests that the gap between GCSE and A Level is bigger in mathematics than in other subjects. Could lack of fluency in algebra be part of the cause of this gap? The Royal Society's Report *Teaching and Learning Algebra pre - 19* (1997) rehearses the arguments for what they call 'the algebra gap' (p16) between GCSE and A Level. One of the options they suggest to overcome this gap is by increased algebra in the pre - 16 curriculum, and indeed since 1998 there has been significantly more algebra in the Edexcel GCSE (the board my pupils sit). The oldest pupils in my sample took their GCSE in June 1998, the youngest will take it this summer, June 2001, but the data still suggests that there is an 'algebra gap'. I researched pupils' *symbol sense* (Arcavi , 1994) which is not just how fluent they are at manipulating symbols, although that is part of it. It includes, amoung other things, an understanding of the power of symbols, the ability to "read" algebraic expressions, the realisation of the constant need to check symbol meanings and a feeling for when to abandon symbols in favour of other approaches. I have already written about some of the teaching implications that emerged from the research (Sharma, 2000). In this paper I detail how symbol sense differed between GCSE and A Level pupils.

Background

The research involved thirty two high attainers from an Independent Girls school in an affluent suburb of London. The sample was composed of eight pupils from each of the top two sets of years 10, 11, and from the A Level pupils in years 12 and 13 (sixth formers). The sample's Key Stage 3 tests were level 7 or 8 (with one level 6), the GCSE grades of year 11, 12 and 13 were A or A*, and A Level grades of year 13 were in the range A to C.

The instrument was based on classic errors and misconceptions and was a ten question questionnaire that the pupils completed individually. The responses were allocated scores and analysed using a spreadsheet but statistical testing with such a small sample was not viable. Follow up semi structured interviews enriched and clarified the data available from the questionnaire.

In this paper I will examine three of the questions from the questionnaire to illustrate the difference in the responses pre and post GCSE.

Question

Some pupils were trying to solve a problem. They ended up with the equation

$$\frac{8y - 6}{4} = \tfrac{1}{2}(4y - 3)$$

Their solution is given below

$$\frac{8y - 6}{4} = \tfrac{1}{2}(4y - 3)$$

$$2y - 6 = 2y - \tfrac{3}{2}$$

$$-6 = -\tfrac{3}{2}$$

They knew their answer was silly but could not spot what they had done wrong. Can you?

Having symbol sense "is at the heart of what it means to be competent in algebra" (Arcavi, 1994 p32). Lee and Wheeler's (1987) study amoung 15 and 16 year olds (GCSE age pupils) found that pupils cancelled part of the numerator with part of the denominator. Would A Level pupils do the same?

Nine sixth formers as opposed to three year 10 and 11 pupils were able to answer the question completely (see table below). The responses to this question not only differed by year group but also by the method used to spot the error. One year 10 and one year 11 pupil factorised and cancelled. Their explanations stated "8*y* - 6 must be treated as one number" and "you can only cancel down multiplications". They were treating 8*y* - 6 as an "object" (Sfard & Linchevski, 1994) and had learned that, with this kind of expression, they should **not** divide, their teachers probably anticipating just the sort of errors which were in the given question! Even though these pupils did not state specifically that the -6 had not been divided by 4 I classified their explanations as 'complete'.

	Years 10 & 11	Years 12 & 13
Incorrect algebra or explanation	4	1
Algebra only partially correct OR incomplete explanation	4	2
No explanation but algebra correctly worked by division or multiplication	3	3
Algebra correct but incorrect explanation	2	1
Complete explanation with or without algebra	3	9

Just Explanation

Three of the sixth formers gave a simple statement by way of explanation compared to only one year 11 and no year 10 pupil. They were able to "read" the written symbols and they were sufficiently competent both to be able to manipulate the symbols mentally and give a brief written explanation. A typical response was

> They have not divided the -6 on the left hand side by 4 in line 2. If they had it would become -$\frac{3}{2}$ and the two sides would be equal (year 13).

Showing the correct division

Just under half of the sixth formers spotted the error by showing how on the second line the -6 should have been divided by 4 as well as the 8*y*, these were classified as giving the ‘complete explanation’.

Some year 10 and 11 pupils also divided but they made mistakes in their algebra, gave incorrect or no explanations. They realised that they should be dividing but, unlike the sixth formers, they were not skilful enough in algebraic manipulation neither was their symbol sense good enough for them to be able to interpret their symbols even when they did divide correctly. An example from one year 10 pupil is

$$\frac{8y-6}{4} = \tfrac{1}{2}(4y-3)$$
$$\frac{4y-3}{2} = \tfrac{1}{2}(4y-3)$$
$$2y - \tfrac{3}{2} = 2y - \tfrac{3}{2}$$
$$= 1$$

Faced with the identity this pupil was perplexed, giving a solution (classified as ‘algebra only partially correct’).

Reworking by Multiplying

Half of the year 10 and 11 pupils but only two sixth formers could only tackle an explanation by reworking the equation from the beginning, usually by multiplying both sides by 4 or 2. The manipulations did not help them spot the error because they either made mistakes themselves (classified as ‘algebra only partially correct’), or else ended up with an identity which they could not interpret (Sharma, 2000) giving ‘no explanation’ or an ‘incorrect explanation’.

Interviews

Although all the pupils were able to state verbally what was wrong with the given question the year 10 and 11 pupils gave their reasons as

> They’ve divided the whole, each thing by 4; they haven’t put the brackets are 8*y* - 4 (year 11)

> They divided the $8y$ - 6 by 4 ..not as a whole unit…they didn't divide the -6. You have to keep the $8y$ - 6 as a whole unit and treat it as one (year 10)

whereas the sixth formers knew that division is distributive illustrating their greater fluency in manipulating the symbols.

Question

Micheala was asked to solve the equation $2y^2 = y$. Here is her solution: $2y^2 = y$ $2y = 1$ $y = ½$ Do you think her solution was the complete correct solution? a partial correct solution? an incorrect solution?

Did the pupils have enough symbol sense to check symbol meanings (Arcavi, 1994) and recognise this as a quadratic equation with possibly two solutions? I expected all the year 13 pupils and most of the year 12 pupils (halfway into the A Level course when they completed the questionnaire) to realise there could be two solutions. My experience as a teacher suggested that the equation in its given form (not equal to zero) would not be recognised by most year 10 and 11 pupils and they would think that the solution was the *complete correct* one.

Nearly half of the sixth formers as opposed to just one year 11 pupil, were able to correctly demonstrate algebraically or by explanation that the solution was *partially correct* (see table below). In this question just two sixth formers and one year 11 pupil were able to answer the question by a simple explanation. The others manipulated the equation correctly, making one side zero, factorising and finding two solutions, hence recognising that the given solution was only *partially correct*.

	Years 10 & 11	Years 12 & 13
Explanation incorrect or missing	4	4
Algebra incorrect	6	2
Algebra or explanation partially correct including those who substituted y = ½	5	3
Completely correct explanation or algebra	1	7

The method of substitution done by three of the year 10 and 11 pupils but only one year 12 pupil to check whether or not the solution was correct was misleading and all four pupils who used this method thought the solution was *completely correct*. Other methods of solving the equation involved division by or subtraction of y, square rooting, and division by 2 all leading to different conclusions (Sharma, 2000). Only two year 12 and 13 pupils used these methods as opposed to six year 10 and 11 pupils. 50% of those whose algebra was incorrect opted for the *completely correct* answer and the other half for the *incorrect* answer. These pupils were manipulating the algebra without a clear objective in mind unlike the pupils whose completely correct replies indicated that they sensed the implications of the y^2 term.

Interviews

The year 10 pupils were confused because it had never occurred to them that a solution could be partially correct; the year 11 pupils thought there should be two solutions because of the power but couldn't find a second solution; two year 12 pupils thought the solution was *completely correct* because they were unable to see the significance of y^2 in the equation. The third year 12 pupil knew that a quadratic could give two solutions but did not seem to recognise this one since it was not equal to zero. When asked she thought a quadratic was

> take x plus with a squared. Or four x squared plus four x plus a constant……its primarily in terms….. um ….. a letter that's been squared.

Only the two year 13 pupils were decisive in their decision to change their minds and put $y = ½$ as a *partially correct solution*. Hence, including those interviewed, just over half of the sixth formers had sufficient symbol sense to know the solution was *partially correct* as opposed to only two year 10 and 11 pupils.

Question

Look at the following

$\dfrac{2x + 4}{x + 2} = 10$		(line 1)
$2x + 4 = 10(x + 2)$		(line 2)
$2x + 4 = 10x + 20$		(line 3)
$4 = 8x + 20$		(line 4)
$-16 = 8x$		(line 5)
$-2 = x$		(line 6)

Is this answer definitely true ? possibly true ? never true ?

State how you know.

Symbol sense would have required the pupils not to rush in and manipulate the equation on the first line (Arcavi, 1994), but to pause, “read” the symbols and notice that $(2x + 4)/(x + 2)$ is always equal to 2 and can never equal 10. I expected this to be a difficult question for all the pupils. It is similar to the only example of symbol sense cited in the Royal Society Report (1997), the implication in the Report being that algebra of this kind is what teachers should be teaching.

Only two year 13 pupils out of the whole sample spotted by inspection (see table below) that the numerator is double the denominator.

Year Group	10	11	12	13
Thinks the algebra is logical or no explanation	4	7	4	3
Substitutes x = -2 and tries to divide 0/0	4	1	3	2
Substitutes x = -2 and realises that this would give denominator zero			1	1
Realises (2x + 4)/(x + 2) = 2 not 10				2

The reasoning very nearly corresponded with the responses (see table below) in that, with the exception of the year 10 pupils, all who thought the algebra was logical ticked *definitely true;* all who substituted thought it was *never true*. The year 10 pupils were the only ones who thought it could be *possibly true*, the interviews eliciting their confusion over this question.

	Year 10	Year 11	Year 12	Year 13
Definitely True	3	7	4	3
Possibly True	3			
Never True	2	1	4	5

That all bar one of the Year 11 pupils working towards their GCSE examination thought the solution was *definitely true* is perhaps an indication that the current assessment system encourages a technique driven curriculum. They reworked the equation, could see nothing wrong with the algebra and since it was a ‘nice’ answer never thought to check, by substitution for example, whether or not the solution was correct.

Interviews

There as a mixed response with two of the year 10 and 11 pupils and the year 12 pupils unsure of what to do with line 1 of the equation like statement. One year 11 pupil was able to “see” that the first line should equal 2 not 10. One year 10 pupil and both the year 13 pupils suggested the first line should be amended to:

$$\frac{10x+20}{x+2}=10$$

One possible explanation as to why this is such a difficult question is perhaps given by a year 13 pupil. She wanted to solve the equation and thought the x's should be changed in some way to x^2 so that she could obtain a value for x. She was asked:

I: Do you think always all equations then to be such that we are going to find a value for x?

13 E: No, its just that you expect them at this level to be able to work out to a particular value. I think its part of the teaching that you are taught. You are given questions where you will have values which will work so to have one where it doesn't work non plusses you because we are always taught that the questions that we are going to see are going to have an answer and the fault is with **us** it they don't work out not with the equation.

Conclusion

I have reviewed above only three out of ten questions. Out of ten questions one did not have a 'right' or 'wrong' answer so the table below shows how many, out of nine questions, were completely correctly answered.

The sample size is too small to be able to come up with any general conclusions, however it is noticeable that only **one** GCSE pupil, who is extremely able, got more than three questions completely correct as opposed to ten A Level pupils.

Number of questions completely correct

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Year 10	3	2	2	1						
Year 11	1	2	3	1			1			
Year 12		1	1	1	2	2		1		
Year 13		1	1	1	2	1	1	1		

The data does not suggest that studying A Level per se gives a pupil symbol sense; there are other factors involved which have not been researched. All the pupils were high attainers but in spite of the GCSE syllabus now including more algebra the study does indicate that the gap in symbol sense between GCSE and A Level includes an inability to select an appropriate method for, and a lack of competency at, manipulating symbols; a belief that following algebraic logic correctly step by step will always result in the 'right' solution; an inability to check symbol meanings with expected outcomes, for example the meaning of x^2 in an equation, and a reluctance to "read" symbolic expressions for reasonableness.

If symbol sense is to be acquired in greater measure by GCSE pupils and the 'gap' reduced then I suggest that teachers need to try to introduce high attainers to algebraic expressions which are not always 'standard' either in format or in outcome. We should pepper our teaching with illustrations of different ways of manipulating algebra rather than teaching a 'rule' which we know is a 'safe' one because it will avoid errors and take every opportunity to discuss expected outcomes with pupils before a problem is solved.

References

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