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ON THE 'TAIL' OF A SEQUENCE, THE UNIVERSAL QUANTIFIER AND THE FORMAL DEFINITION OF CONVERGENCE

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Abstract: *Achieving logical coherence and understanding of the precise meaning of a symbol are crucial points in the learning of a first year mathematics undergraduate. Here, in the context of a course in Linear Algebra and Calculus, we examine students' written work in order to address issues regarding the understanding of the formal definition of convergence for numerical sequences. In particular we focus on the students' use of the universal quantifier,* $\forall$*, in ways that suggest that the students may neglect the 'universality' in its meaning: in their applications of the definition of convergence, in some occasions, not all* ε *are covered and, in some others, a finite number of the terms of the sequence, are left out of the argument.*

This study originates in a collaboration between the School of Mathematics (where the first author teaches first year undergraduates) and the School of Education (where the second author is a lecturer) in UEA. The study is funded by the Nuffield Foundation and is carried out in two phases. The first phase, regarding Calculus and Linear Algebra, lasted three months (October-December 2000). The second phase, regarding Probability Theory, is now in progress and will also last three months (January- March 2001). For the methodology and aims of this study see Note at the end of the paper.

The concept of convergence of sequences and series is amongst the 'oldest' advanced mathematical concepts that have raised research interest - as suggested in reviews such as (Kaput & Dubinsky 1994) and (Tall 1992)). The concept of the limiting process is central in most of these reviews as its understanding impinges upon a complex network of ideas and an equally complex novel notation. Symbols such as ε, δ, $\exists$, and $\forall$ 'interfere' heavily with the learners' building up concept images from the definition. Other 'interferences' include those from graphical representations of the limiting process - for example the one of the limiting procedure as a staircase: these have been reported as prone to evoking ambiguous perceptions of the infinitely recurring process lying at the heart of the concept.

In this paper we wish to extend the discussion on some of these interferences - mostly those coming from students' interpretation of the universal quantifier. In some of our discussion we also examine the influence of a certain graphical representation of the limiting process.

The first problem sheet question we wish to discuss is Question 4.4, from the fourth cycle of data collection:

(4)* (a) Write out carefully the exact meaning of the statement "the sequence (a_n) converges to A as $n \to \infty$".

(b) Prove using (a) that the sequence $a_n = 2 + \frac{1}{\sqrt{n}}$ converges to 2.

(c) Prove using (a) that the sequence $b_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$ converges to 2.

The answer suggested in the notes written for the tutors and the students by the setter of the question was:

4. (a) "the sequence (a_n) converges to A as $n \to \infty$" means

$$\forall \epsilon > 0, \exists N \text{ such that } n \geq N \Rightarrow |a_n - A| < \epsilon.$$

(b) Given any $\epsilon > 0$, choose $N = \frac{1}{\epsilon^2} + 1$. Then

$$n \geq N \Rightarrow n > \frac{1}{\epsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \epsilon.$$

On the other hand,

$$|a_n - 2| = \frac{1}{\sqrt{n}},$$

so we have shown that

$$n \geq N \Rightarrow |a_n - 2| < \epsilon,$$

which proves that a_n converges to 2 as $n \to \infty$.

The intentions of the question setter appear to be as follows: Question 4.4 is an exercise in handling new definitions, using them on examples, in particular, an exercise in the use of quantifiers (these had just been introduced to the students in the lectures).

The students' responses to the first part of the question were of two types and both regarded the interpretation of the words 'exact meaning'. Students understood this either as a request to reproduce the quantified statement of the formal definition or as a request to describe their 'image' of convergence. We perceived this as a case of the overt interplay between concept images and concept definition (Tall and Vinner 1981), here in the context of interpreting and engaging with a mathematical task. Our focus here will be on several typical student responses to part b of Question 4.4 and, in particular on the students' rather problematic choice of N in the application of the definition of convergence. Contradicting the inequality $|a_n - A| < \varepsilon$ in the definition, most students chose N in such a way that this modulus can become equal to ε. For example, Nicolas:

(4) a) $\forall \varepsilon > 0 \; \exists N$: if $n \geq N$, $|a_n - A| < \varepsilon$

b) $a_n = 2 + \frac{1}{\sqrt{n}} \rightarrow 2$

side calculation

$|a_n - 2| = \frac{1}{\sqrt{n}}$

$\frac{1}{\sqrt{n}} < \varepsilon$

$n > \frac{1}{\varepsilon^2}$

Given any $\varepsilon > 0$, choose $N = \frac{1}{\varepsilon^2}$

then if $n \geq N$ then $|a_n - 2| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}} < \varepsilon$

Here *N* needs to be chosen strictly greater than $1/\varepsilon^2$ (or equal to the closest integer to $1/\varepsilon^2 + 1$ as suggested in the proposed answer). One underlying problem here could be the interpretation of $\forall$. Students seem to think that if some of the ε are left out, this is not a problem; that the property has to be true 'for all ε' but, if it is true for most of them, the statement is still valid. Of course, this hurdle could be overcome with a rearrangement of the choice of ε (so that the case for the equality is accommodated) but none of the students suggested such a procedure. It is noteworthy that this problematic application of the definition of convergence appeared also in the responses of the students who had reproduced the definition in part a accurately. We report similar gaps and inconsistencies also in (Nardi and Iannone, submitted).

Two weeks later another application of the formal definition of convergence, Question 5.6, from the fifth cycle of data collection, appeared in the problem sheet:

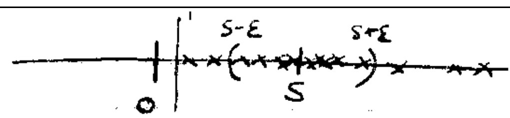
(6) Write out a careful proof of the following useful lemma sketched in lectures. If $\{b_n\}$ is a positive sequence (for each n, $b_n > 0$) that converges to the number $s > 0$, then the sequence is bounded away from zero: there exists a number $r > 0$ such that $b_n > r$ for all n. (Hint on how to start: since $s > 0$, you may take $\frac{1}{2}s = \epsilon > 0$ in the definition of convergence.)

The suggested answer was:

6. Let $\epsilon = \frac{s}{2} > 0$ in the definition of convergence. Then there is an N such that $n > N \Rightarrow |b_n - s| < \frac{s}{2} \Rightarrow b_n \geq \frac{s}{2}$. Then, for any n, $b_n \geq r = \min\{b_1, \ldots, b_N, \frac{s}{2}\}$, which is the minimum of finitely many positive quantities, hence is positive.

In the Question Clinic held before the homework was due in (observed by the first author), to the students' request, the lecturer explained the hint given and what happens when $\varepsilon = s/2$ is chosen. What he didn't say was that this choice deals with the 'tail' of the sequence, but the terms before N need to be dealt with as well. However, to support the students' intuitive understanding, he also drew a diagram to show where the end tail of the sequence, from term b_N onwards for the case $\varepsilon = s/2$, was falling, namely in the interval (s-s/2 , s+s/2). This diagram was variably understood by the students. Here is, for example, Hazel's response:

6.



b_n converges on s
$b_n > 0$
$b_n > r \quad \forall n$

Want to show that $\exists r, \forall n : b_n > r$

Choose $\varepsilon = \frac{1}{2}s$

then $\exists N$ such that $n \geq N$

$\Rightarrow |b_n - s| < \frac{1}{2}s$

$s > 0 \quad b_n > 0$ $\qquad b_n < \frac{1}{2}s + s$ $\qquad$ and $\qquad b_n < s - \frac{1}{2}s$

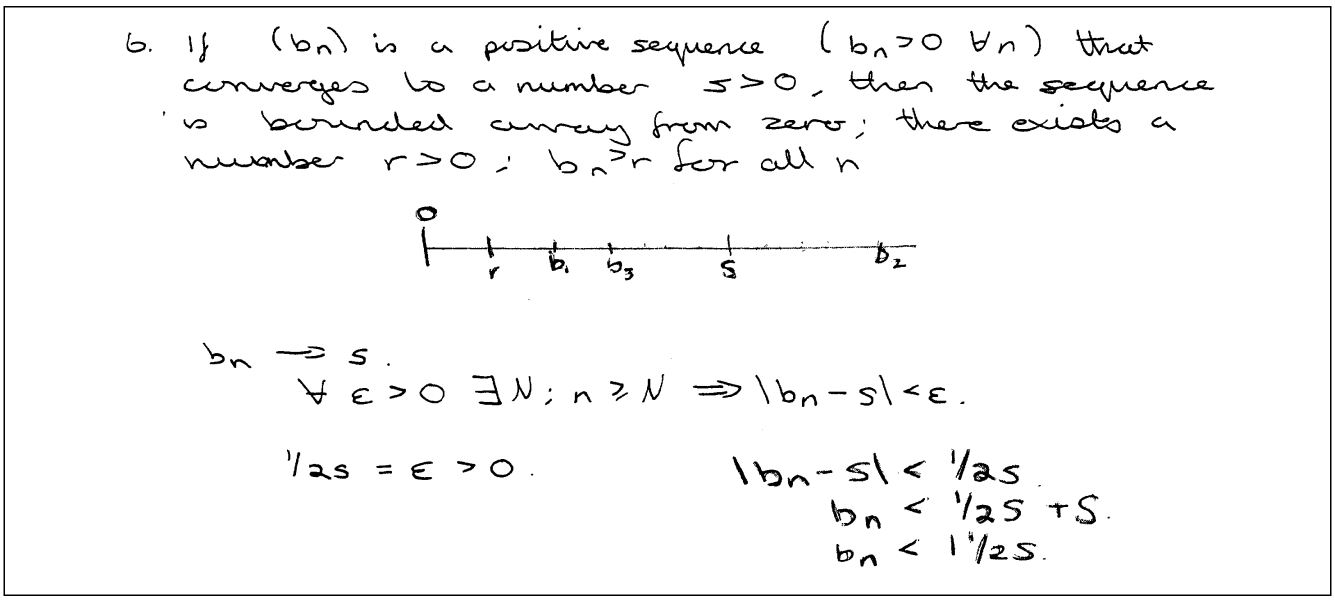
$b_n < \frac{3}{2}s$ $\qquad b_n < \frac{1}{2}s$

↓ upper boundary $\qquad$ ↓ lower boundary

$\therefore r = \frac{1}{2}s$

Hazel copies the diagram used by the lecturer in the Question Clinic, but fails to observe that there are 'little crosses', namely terms of the sequence, which fall outside the interval (*s*-*s*/2 , *s*+*s*/2) and need to be considered as well. Disregarding the $\forall n$ requirement, she then proposes as *r* the extreme of this interval, that is *s*/2.

While Hazel casually contradicts the diagram she copied from her Question Clinic notes, another student, Emma produces a diagram, arguably originating in the one she saw in the Question Clinic, albeit one that is neatly arranged to fit her argument:


6. If (b_n) is a positive sequence $(b_n > 0 \ \forall n)$ that converges to a number $s > 0$, then the sequence is bounded away from zero; there exists a number $r > 0$; $b_n > r$ for all n

$b_n \rightarrow s$.

$\forall \varepsilon > 0 \ \exists N ; n \geq N \Rightarrow |b_n - s| < \varepsilon$.

$\frac{1}{2}s = \varepsilon > 0$.

$|b_n - s| < \frac{1}{2}s$

$b_n < \frac{1}{2}s + s$.

$b_n < 1\frac{1}{2}s$.

In her diagram Emma places all the terms of the sequence after *r*, the number that she claims she aims at finding in her immediately preceding sentence (note that *r* does not appear in Hazel's diagram but appears in her writing). In a sense, *r* in Emma's diagram is what she wants it to be but not what it turns out to be in her writing. Emma also places various terms of the sequence around *s*, the limit of the sequence, but her diagram does not seem linked with her subsequent application of the formal definition of convergence (a characteristic commonly observed in the relevant literature): for example, there is no ε and no *s*/2 in her diagram. Despite their differences in degree of association between diagram and writing, Emma and Hazel, produce the same incomplete coverage of b_n: like Hazel, Emma leaves out of her argument the terms of the sequence before N. Again, the $\forall n$ requirement in the exercise has been disregarded.

Not all students resorted to the diagram used by the lecturer in the Question Clinic. However they all produced similarly incomplete arguments. Consider, for example, Wayne's response:

6) $\{b_n\}$ is +ve $\forall n$ $b_n > 0 \longrightarrow s > 0$

prove $\exists r > 0$ such that $b_n > r \ \forall n$.

Consider definition of convergence for this sequence:

$\forall \epsilon > 0, \exists N : n \geq N, |b_n - s| < \epsilon$

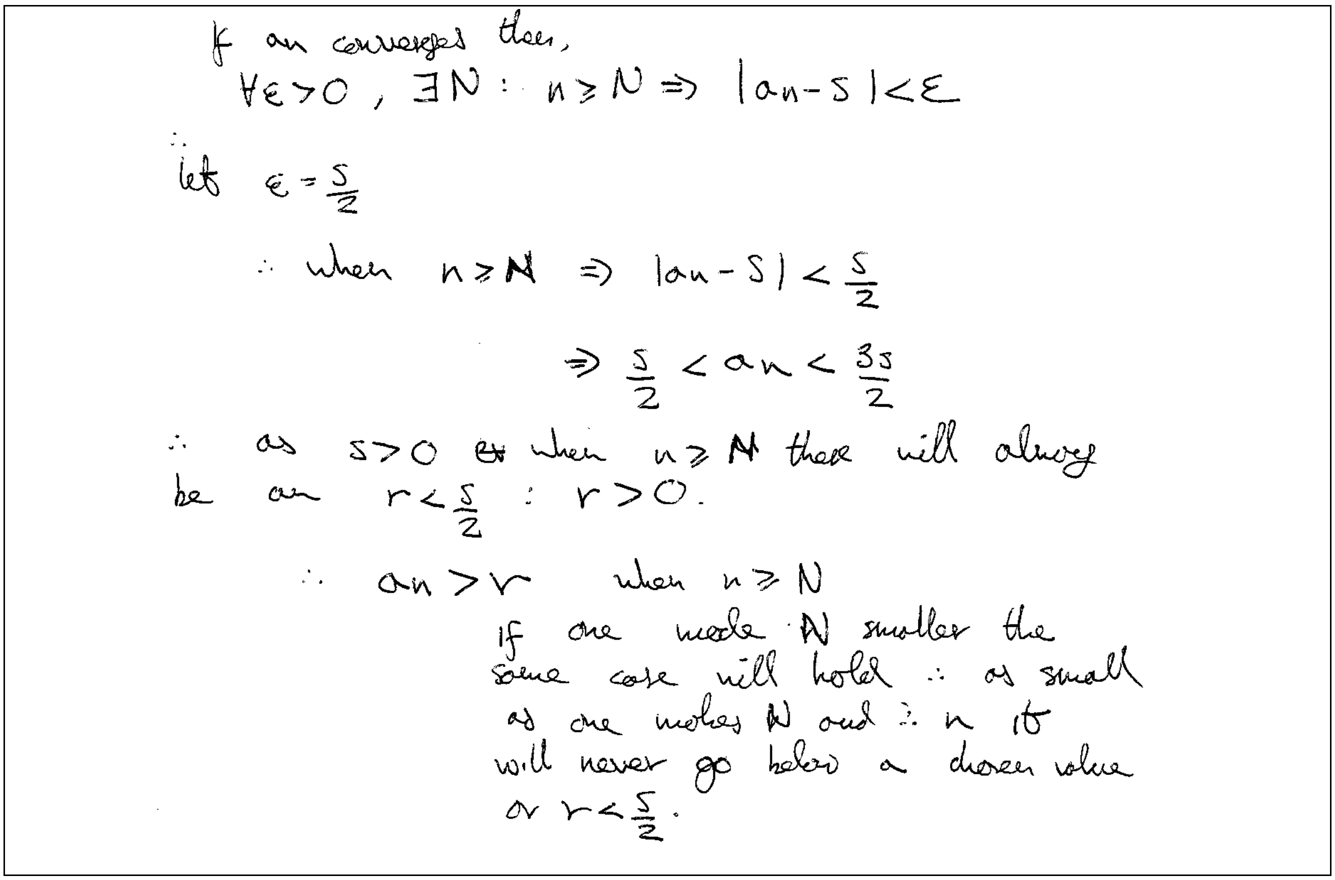
take $\epsilon = \frac{1}{2}s > 0$

hence $|b_n - s| < \frac{1}{2}s$ ie $\frac{1}{2}s < b_n < \frac{3}{2}s$

$\therefore \exists r (= \frac{1}{2}s) > 0$ such that $b_n > r \ \forall n$.

The inconsistencies in Wayne's argument lie only a couple of lines away from each other: he states the formal definition of convergence for a general ε, applies it for $s/2$, rearranges its central inequality so that the modulus || is removed and the statement begins to resemble the desired $b_n>$... one and then, once he sees that b_n appears greater than something, $s/2$, he rushes into calling this r and claims that his task is accomplished. It appears that the $\forall n>N$ condition in the definition of convergence is so packed in, it is such a compressed component of a statement that the student is conditioned to reproduce when required, that it is not meaningful to the student - or, at least, not meaningful enough to prevent him from contradicting himself in the next two lines of his written argument.

Amongst the sixteen students whose written work we analyse, only one appeared to observe that the hint suggested by the lecturer in the problem sheet was only a hint on 'how to start' and that there was further action to be taken towards completing the task. This was Joseph:



If a_n converges then,
$\forall \varepsilon > 0,\ \exists N : n \geq N \Rightarrow |a_n - s| < \varepsilon$
∴ let $\varepsilon = \frac{s}{2}$
∴ when $n \geq N \Rightarrow |a_n - S| < \frac{s}{2}$
$\Rightarrow \frac{s}{2} < a_n < \frac{3s}{2}$
∴ as $s > 0$ & when $n \geq N$ there will alway be an $r < \frac{s}{2} : r > 0$.
∴ $a_n > r$ when $n \geq N$
if one made N smaller the same case will hold ∴ as small as one makes N and ∴ n it will never go below a chosen value or $r < \frac{s}{2}$.

Joseph states the formal definition of convergence for a general *ε,* applies it for *s*/2, removes the modulus || and the statement begins to resemble the desired a_n>... one. He thoroughly observes that *s*/2 is a candidate for *r* as it is greater than 0 and is a bound for a large number of the terms of the sequence, its 'tail'. Crucially he maintains the expression 'when *n*>*N* ' throughout his argument and thus allows himself to observe that his argument needs to accommodate the terms before *N*. Unfortunately his effort is severely curtailed by his understanding of the nature of *N*. For every *ε,* hence for *s*/2, the choice of *N* is a fixed and unalterable action. Joseph however, in order to cover a_n for n before N, attempts to present an argument according to which N can be taken 'as small as one wants'. This, he claims, would make $s/2$ a suitable r. But N cannot change, therefore Joseph's claim that $r = s/2$ is an insufficient one. Its deficiency seems to be due to his neglecting the existential quantifier that defines N.

In a sense then Joseph's response, despite its sophistication in observing the need for an argument that covers $a_n\ \forall n$, whether before, on or after N, suffers from a weak understanding of a quantifier that is similar to his peers - of whom we have seen the work of Hazel, Emma and Wayne. His limitations may concern the understanding of the existential quantifier and theirs may concern the understanding of the universal quantifier.

The presence of the idea that it is valid to think of a statement as true if it holds for most, or nearly all, cases should not be surprising. It has been part of the students' mathematical practice throughout most of their years of schooling. Estimates, experimentation with a finite number of cases and inference from this process is common practice. Therefore, to the students, complete coverage of all cases has not been an issue worthy of substantial attention. Of course, in the evidence we examine in this paper, complete coverage of all terms of the sequence may not have been pursued by the students simply because - for the reasons we have suggested within each example - they did not notice that their argument did not cover all b_n. But doesn't this not-noticing suggest a lack of alertness regarding the coverage of all cases?

From a pedagogical point of view, this lack of alertness is arguably an outcome of the students' lack of experience with studying examples where the emphasis is on coverage of all cases as truly crucial to the completeness of a solution. In fact at more advanced stages in Analysis and Measure Theory, the rules do appear to bend again when statements appear as true 'for all but a finite number of terms' or 'for all but a set of terms with measure equal to zero'. Of course these statements are made while the relevant proofs accommodate for these 'but...' parts of the argument; in other words, these statements are made with transparency. After rehearsing these arguments quite a few times, their repetition becomes redundant and the expression 'for all but a finite number of terms' acquires an extra level of meaning: if one is using it, then, if challenged, one is able to produce an argument that accommodates this 'finite number of terms'. Therefore it seems that to be 'allowed' the use of such ostensibly not fully accurate statements, to be allowed this compression, one needs to master the art of transparency first. And this sounds like a task that the teaching needs to contribute to.

NOTE

This is the first phase of an Action Research Project (Elliot 1991). It was conducted in 6 cycles of Data Collection and Processing following the fortnightly submission of written work by students during a 12-week term. Within each 2-week cycle students attend lectures and problem sheets are handed out; students participate in Question Clinics, a forum of questions from the students to the lecturers; students submit written work on the problem sheet; students attend tutorials in group of six and discuss the now marked work with their tutor. The two authors then engage in an analysis of the student’s written work. This consists initially of a table that includes a brief description and interpretation of the students' responses to the questions in the problem sheets: each row corresponds to a question in the problem sheet and each column to a student. The penultimate column collates informal comments on each question offered by the course lecturer and seminar leaders. The final column collates the researchers' preliminary interpretations regarding the students' difficulties with each question. The contents of this final column constitute the basis of the Analytical

Account, a text that highlights major problematic areas in the students' writing, links to relevant literature and suggests points of caution and action for the tutors (for more details see (Nardi and Iannone 2000)).

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