Winter, J. (Ed.) (Proceedings of BCME5) Proceedings of the British Society for Research into Learning Mathematics 21(2) July 2001

INDICATORS OF ABSTRACTION IN YOUNG CHILDREN'S DESCRIPTIONS OF MENTAL CALCULATIONS

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Abstract: *The National Numeracy Strategy encourages class discussion of mental calculation strategies but when children respond to "How did you work that out?" teachers may be able to learn more than just what strategies were employed. This paper proposes three 'indicators of abstraction' which might provide indications of childrens developing conceptualisations. The indicators related to children's descriptions of their calculations are 'metaphor', 'expression of generality' and 'method'. Each has three categories of abstraction. Data drawn from a study with 7 to 9 year old children suggests that responses categorised as 'representative' and 'symbolic' are associated with high levels of accuracy whilst lower accuracy is more often associated with higher frequency of 'concrete' responses.*

Introduction

The children in this study took part in very similar classroom activities yet gave qualitatively different responses to interview questions requiring mental calculations. This paper addresses two questions "What are the indicators of different conceptualisations for mental calculations?" and "How are these indicators related to achievement?"

The data presented was collected as part of an investigation into the influences of classroom activities on pupils' language related to mental calculation. Preliminary studies (Bills 1999, 2000, in press) suggested that pupils often used language related to physical manipulation of classroom materials or language related to written algorithms when talking about mental calculations. This use of the language of one context to describe another is referred to as 'metaphor'. The metaphors may be related to manipulation of objects, to number lines or to manipulation of symbols. Pupils' descriptions of a calculation just performed also show three levels of 'generality'. Some pupils describe what they did with the numbers in the calculation whilst others use the numbers to explain their procedure. Others simply give a general rule. The 'method' used also falls into three categories. Counting is the least sophisticated and symbol manipulation the most abstract. Between these two are methods which retain a sense of the size of the numbers but use manipulation strategies.

Thus each of the 'indicators', metaphor, generality and method, has three levels of 'abstraction' termed 'concrete', 'representative' and 'symbolic'. The data presented shows that the most accurate pupils frequently use the more abstract modes of response. This leads to the conclusion that the higher levels of response indicate a greater facility with numerical procedures and thus can indicate that pupils are progressing toward procedural competence.

Abstraction

> "I take it as given that all cognition, whatever its nature, relies upon representation, how we lay down knowledge in a way to represent our experience of the world"
>
> (Bruner, 1996, p95)

Bruner (1966) suggests that there are three modes of representing our experiences of the world: enactive (related to action), iconic (related to physical or mental image, independent of action) and symbolic (related to language and other symbol systems). Furthermore he suggests that, whilst there are changes of emphasis that occur with development of representation, the interplay of the three persists as a feature of adult intellectual life.

Paivio (1971) insists that it is "simply asserting a truism" (p18) to say that modes of representation evolve within the individual from the more concrete to the more abstract. Piaget's model of cognitive development assumed a process of adaptation that is not predetermined but arises out of external action on physical objects and with other people (Richardson, 1998). Early 'sensori-motor' schemata are behaviour patterns that co-ordinate sensation and action and they need not involve any mental representation. The development of internal representations makes thought and language use possible. Piaget assumed that humans have a natural tendency to organise knowledge into coherent cognitive systems. Generalisation and categorisation resulting from 'abstraction' from our experiences are thus seen as the mechanisms for cognitive development. Piaget distinguished 'empirical abstractions', when sensory patterns that recur in a number of experiences are retained and co-ordinated to form concepts, from 'reflective abstraction' which results from mental operations on concepts (von Glasersfeld, 1995).

Harel and Tall (1991) refer to 'Generic abstraction' which occurs when specific examples are seen as typical of a wider range of examples embodying an abstract concept. Balacheff (1988) defined a 'generic example' as an object that is not there in its own right but as a characteristic representative of the class. For Mason and Pimm (1984), also, the use of generic examples marks a step toward a statement of generality but it is expressed in terms of the particular. There is a suggestion here that particular instances may both lead to generalisation and provide a means of expressing generality. This could be true for both concepts and procedures.

Gray & Tall (1994) note the dichotomy between things to do (procedures) and things to know (concepts) so that procedural and conceptual learning are both requisites of mathematical learning. In mathematics learning the formation of a mental object, abstracted from a procedure, which can become the subject of thought and itself be used in more abstract processes, is fundamental to mathematical thinking. Gray & Tall (1993) argue that those who fail at mathematics have failed to progress satisfactorily from the procedures of counting to the processes of arithmetic and similarly fail to generalise from other learned procedures in other areas of mathematics. They distinguish between flexible thinkers, for whom a symbol is a mathematical object that can be manipulated in the mind, and instrumental thinkers for whom the symbol signifies a procedure to be carried out. They argue that those who are unsuccessful in mathematics are doing a more difficult mental task in trying to use a mental analogue of the procedure.

Recognition of generality lies at the heart of mathematics. The use of language associated with generality may be an important clue to the state of children's mental representations. Lakoff & Johnson (1980) argue that communication is based on the same conceptual system that we use in thinking and acting so that language is a source of evidence for what the system is like. They suggest also that the conceptual system is fundamentally metaphoric in that we think of concepts in terms of others. Thus the language used indicates the experiential basis for our conceptualisations. In the context of this study language associated with counting of objects, for instance, can indicate that pupils have been influenced in their thinking by those activities.

Method

Lesson observations and pupil interviews were first conducted with two classes from Year 3 in a school for children aged 5 to 11 years in a large middle-income village near Birmingham, from September 1998 to July 1999. The same pupils were observed and interviewed in the following year. The 80 children in the year group had been placed in one of three sets for Mathematics based on their previous attainments. Lessons with the high attainment and the middle attainment groups were observed and a sample of 14 pupils from the first and 12 from the second was interviewed in December, March and July in each year. The samples were chosen to represent the spread of achievement levels in each group.

There were 45 mental calculation questions in total over the 6 interviews, each presented orally and pupils had no pen, paper or materials to assist them.

Type	Description	Examples of questions
1	1-digit addend	17 + 8, 17 + 9 (repeated in each interview)
2	Missing addend	13 + * = 18, 30 + * = 80, 27+*=65
3	2-digit addition	48 + 23 (repeated in each interview)
4	Addition of multiple of 10	97 + 10, 597 + 10, 1097 + 10, 1197 + 10

5	Counting	What comes before 380, 2380, 12100; after 12386
6	Rounding	Round 2462 to the nearest ten, 239 to nearest hundred
7	Recent topic	What is difference between 27 and 65, 0.6+0.7
8	Recent topic	65 subtract 29, Read time (11:40), 0.1 times by 10
9	Division and fractions	quarter of 40, third of 48, 140 divided by 3
10	Multiplication	48 multiplied by 3, 47 multiplied by 5

After the calculation pupils were asked "What was in your head when you were thinking of that". Pupils talked about how the calculation was performed and these responses were categorised for metaphor, generality and method. Analysis was aided by the use of a Filemaker Pro database.

Metaphor categories

Metaphors were categorised in accord with Lakoff and Nunez (1997) 'grounding metaphors':

category 1 — 'object collection' uses the language of manipulation of concrete objects and counting.

category 2 — 'arithmetic as motion' numbers are represented as distances and positions, operations as movement.

category 3 — 'object creation' is the most abstract in treating numbers as objects using a language of manipulation of symbols.

Examples of responses to "30 add something is 80. What is the something?"

collection — just got my hand like and just added 30, 40, 50, 60, 70, 80

motion — 40 add 40 is 80 but it goes one ten down so I have to put a ten up

creation — I was thinking of 3 add something equals 8

Characteristics of the categories were as follows:

collection — counting, including counting tens. Use of "add", "take", "more", "gives", "with".

motion — talk of number lines, words related to motion and position such as "go", "up", "down", "forward".

creation — place value language, derived facts and known facts. Use of "is", "equals", "make", "sums", "the", "it".

Generality categories

Pupils' 'expressions of generality' in their responses to calculation questions were categorised as:

category 1 — 'particular' when they talked about the specific numbers given.

category 2 — 'generic' when the numbers served to illustrate a procedure.

category 3 — 'general' when they gave a general rule with little mention of the numbers given in the question.

Examples of responses to "17 add 9" were:

particular — I added the 5 and then I just added 4 on

generic — 9 is nearly 10, so you add that onto the 17 ... then you remember you had 9 instead of 10, so you take one off

general — make the 9 to 10, add a ten and take one away

Method categories

The methods children employed for calculation were categorised as:

category 1 — 'concrete' when they counted.

category 2 — 'holistic' when the sense of the size of the number was maintained though procedures were followed.

category 3 — 'symbolic' when separate-digit methods similar to written algorithms were used.

Examples of responses to "97 add 10"

concrete added 5 on then added another 5 on and see what I got

holistic you keep the 7, 90 add 10 is a hundred, add the 7

symbolic you just use 9 and then go to ten then just a zero but put ten's zero to the 7

For convenience the categories for each of the indicators are referred to as concrete, representative and symbolic.

Results

The number of correct answers, out of 45, had a range from 13 to 38. The mean was 28 and st. dev 7.4. There were 4 'high-accuracy' pupils who scored more than 1 sd above the mean and 5 'low-accuracy' pupils who's score was less than 1 sd below the mean. The rest were regarded as 'middle-accuracy'. The number of responses in each category for each indicator for these groups were counted. The differences in distributions of categories for each group of children were statistically significant ($p<0.005$).

The differences in metaphor use was quite marked. The lowest achievers were much more likely to use metaphors related to counting and manipulation of objects:

	Number of responses with percentages of row totals in bold							
	Metaphor 1		Metaphor 2		Metaphor 3		Totals	
High-accuracy	66	**36**	18	**10**	101	**55**	185	**100**
Middle-accuracy	179	**30**	95	**16**	321	**54**	595	**100**
Low-accuracy	71	**49**	23	**16**	51	**35**	145	**100**
Totals	316	**34**	136	**15**	473	**51**	925	**100**

The methods used also illustrate that the lowest achievers rely most on counting methods. The highest achievers use the most holistic methods and very infrequently use counting .

	Number of responses with percentages of row totals in bold							
	Method 1		Method 2		Method 3		Totals	
High-accuracy	19	**11**	84	**47**	74	**42**	177	**100**
Middle-accuracy	109	**19**	240	**42**	228	**40**	577	**100**
Low-accuracy	54	**43**	37	**30**	34	**27**	125	**100**
Totals	182	**21**	361	**41**	336	**38**	879	**100**

These two tables taken together illustrate that when the highest achievers use metaphors of object collection they do so in the context of holistic and symbolic methods. Not in the context of counting. The most striking difference however is in the expressions of generality in the responses:

	Number of responses with percentages of row totals in bold							
	Generality 1		Generality 2		Generality 3		Totals	
High-accuracy	59	**32**	89	**48**	36	**20**	184	**100**
Middle-accuracy	191	**33**	305	**52**	87	**15**	583	**100**
Low-accuracy	81	**60**	40	**30**	14	**10**	135	**100**
Totals	331	**37**	434	**48**	137	**15**	902	**100**

The important point here is that the low achievers, irrespective of method used, are much more likely to simply describe what they had done. The higher achievers are more likely to use the numbers to explain a procedure and the highest achievers most often simply state general rules.

When responses which followed correct answers are compared with those following incorrect answers it becomes clear that the indicators give indications of procedural competence not simply characteristics of high and low achievers. The style of metaphor and the method described are not strongly associated with the accuracy of response though there is again a slight (but not statistically significant) tendency for correct answers to be associated with higher categories and lower categories with incorrect answers:

	Number of responses with percentages of column totals in bold							
	Metaphor 1		Metaphor 2		Metaphor 3		Totals	
Right	203	**64**	95	**70**	324	**68**	622	**67**
Wrong	113	**36**	41	**30**	149	**32**	303	**33**
Totals	316	**100**	136	**100**	473	**100**	925	**100**

	Number of responses with percentages of column totals in bold							
	Method 1		Method 2		Method 3		Totals	
Right	119	**65**	247	**68**	248	**74**	614	**70**
Wrong	63	**35**	114	**32**	88	**26**	265	**30**
Totals	182	**100**	361	**100**	336	**100**	879	**100**

The same trend was apparent in the associated type of generalisation but the difference in distributions was statistically significant (p<.005). Those who gave a generic or general response were more likely to be correct and those who expressed themselves in particular terms were more likely to be wrong :

	Number of responses with percentages of column totals in bold							
	Generality 1		Generality 2		Generality 3		Totals	
Right	184	**56**	324	**75**	107	**78**	615	**68**
Wrong	147	**44**	110	**25**	30	**22**	287	**32**
Totals	331	**100**	434	**100**	137	**100**	902	**100**

The distinction between high-accuracy and low-accuracy pupils in terms of their expressions of generality is emphasised when their responses for correct and incorrect answers are tabulated. For high-achievers expressions of a generic or general type are more emphatically associated with correct answers:

	Number of responses with percentages of column totals in bold							
	Generality 1		Generality 2		Generality 3		Totals	
Right	39	**66**	79	**89**	33	**92**	151	**82**
Wrong	20	**34**	10	**11**	3	**8**	33	**18**
Totals	59	**100**	89	**100**	36	**100**	184	**100**

When these pupils have given a correct answer they are much more likely to express themselves in a non-particular (generic or general) way. This higher level expression of generality is associated with successful use of the procedure described. For low-accuracy pupils the use of these expressions is not necessarily associated with ability to use the procedure accurately:

	Number of responses with percentages of column totals in bold							
	Generality 1		Generality 2		Generality 3		Totals	
Right	37	**46**	19	**48**	6	**43**	62	**46**
Wrong	44	**54**	21	**53**	8	**57**	73	**54**
Totals	81	**100**	40	**100**	14	**100**	135	**100**

This table suggests that pupils who are inaccurate in mental calculation may have acquired the general rule in the sense that they can express it but are as likely to be wrong as right. They may know the rule but are unable to use it as reliably as the more accurate pupils. They are much more likely to talk only of what they did with the particular numbers but even this is not associated with higher success.

Discussion

The indicators of abstraction provide a means of classifying pupils' descriptions and explanations of their mental calculation procedures. What emerges from the analysis of this sample is that high-accuracy pupils distinguish themselves most by the level of generality they choose to use when explaining what they have done. They are more likely than other pupils to give a general rule when asked "What was going on in your head?" this suggests that they have abstracted the rule from their classroom experiences and that they find it a convenient means of efficiently communicating their procedure. Low accuracy is more often associated with descriptions of what was done with the particular numbers.

The pupils' use of metaphor is not such a good indicator of competence with calculation. Children show very similar proportions of correct and incorrect answers whether they use language associated with object collection, motion or object creation. The method employed shows a slight association with accuracy in that a higher proportion of responses involving single-digit methods are correct than counting methods. Low-accuracy pupils do, however, more often use language and methods associated with counting. This seems to indicate that they have been most influenced in their thinking by their concrete activities whilst the high-accuracy pupils have been more influenced by their activities with symbols.

All these children have been present for the same classroom activities yet have indicated by their language that they have conceptualised from their experiences in different ways. High accuracy pupils demonstrate their competence in the more abstract modes of explaining their procedures. The least accurate pupils most often only describe what they have done. It can be argued that the children could have qualitatively similar conceptualisations yet simply choose different modes of communication. Even if this is the case then those who choose to express themselves in non-particular terms seem to have an advantage over those who express themselves in particular terms. There is thus more to be learned from young children's modes of expression of their calculation procedures than simply what strategy they have employed.

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