DOMAIN, CO-DOMAIN AND RELATIONSHIP: THREE EQUALLY IMPORTANT ASPECTS OF A CONCEPT IMAGE OF FUNCTION

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Students' concept images of function often lack in understanding the concept as inextricably connected with its domain, co-domain and relationship. The impact of this on understanding properties such as 1-1 can be dramatic. Drawing on students' responses to a task involving the exploration of whether five given functions were 1-1, onto, both or neither, I discuss the following distinctive elements in their responses: ***A.*** *use of graphs to identify properties of the function (from relying completely on the graph for mere identification of the property through to substantiating what the graph suggests with verbal explanations and uses of B and C)* ***B.*** *problematic uses of the formal definitions of 1-1 and onto* ***C.*** *effective use of properties specific to the functions in question (e.g. uniqueness of cubic root). The emphasis here is on B.*

Studies of perceptions of function that focus on difficulties the students encounter when they are asked to decide whether a given relationship or a graph represents a function (e.g. Barnes 1988) date back in the 1980s when this concept became a well-studied area of upper secondary and tertiary mathematics education. In these studies (e.g. Dubisnky and Harel 1992), most students do not associate function with its Bourbaki definition - a correspondence between two sets which assigns each element of the first set to an element of the second set: when responding to mathematical problems students usually call upon concept images of function such as a formula, an algebraic relationship, an equation. As observed by Markovits et al (1986) this image is so dominant that it seems to exist at the expense of the equally important notions of domain and co-domain. Here I extend this discussion to include the study of properties of functions such as onto and one-to-one.

The evidence on the learning of functions on which I draw here originates in a collaborative study conducted at the School of Education and the School of Mathematics (where my Research Associate, Dr Paola Iannone, teaches first year undergraduates). The study is funded by the Nuffield Foundation and is carried out in two phases. The first phase, regarding Calculus and Linear Algebra, lasted three months (October - December 2000). The second phase, regarding Probability Theory, is now in progress and will also last three months (January - March 2001). For the methodology and aims of this study see Note at the end of the paper.

The mathematical question that I wish to discuss here (Question 2.3) was included in the problem sheet of Cycle 2 - fourth week of the students' first semester in the course:

3 For each of the following functions $\mathbb{R} \longrightarrow \mathbb{R}$ decide whether it is one-to-one, onto (or both, or neither). Give brief explanations for your answers.
(i) $f_1(x) = \sin x + \cos x$
(ii) $f_2(x) = 7x + 3$
(iii) $f_3(x) = e^x$
(iv) $f_4(x) = x^3$
(v) $f_5(x) = x/(1+x^2)$.
Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is onto but not one-to-one.

In his notes to the students and tutors for the course, distributed after the students' submission of their written responses, the lecturer suggests the following answers:

3)(i) As $\sin x \leq 1$ and $\cos x \leq 1$ for $x \in \mathbb{R}$, we have $f_1(x) \leq 2 \;\forall\, x \in \mathbb{R}$. Thus f_1 is not onto. Also, $f_1(0) = f_1(2\pi)$, so f_1 is not one-to-one.
(ii) For every $y \in \mathbb{R}$ there is a unique $x \in \mathbb{R}$ with $f_2(x) = y$, namely $x = \frac{1}{7}(y-3)$. Thus f_2 is one-to-one and onto.
(iii) Not onto (as $e^x > 0$ for all $x \in \mathbb{R}$): one-to-one (if $y \in \mathbb{R}$ the only real solution to $e^x = y$ is $x = \ln y$).
(iv) One-to-one and onto (a bijection): any real number has a unique real cube root.
(v) Neither one-to-one nor onto: $f_5(1/2) = f_5(2) = 2/5$, so not one-to-one. Also, $f_5(x) \leq 1$ (as this is equivalent to $x \leq 1 + x^2$, and $1 - x + x^2 = (x - 1/2)^2 + 3/4 \geq 0$.

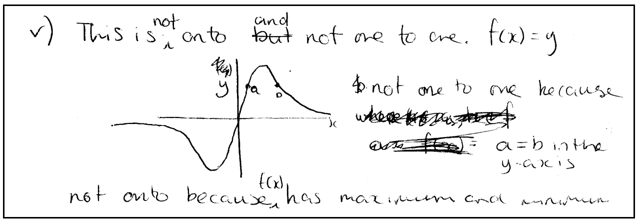
Remark: You might like to think about what bits of calculus can be used to justify more fully the fact that the functions in (iii) and (iv) are one-to-one.
Last part: $f(x) = x(x-1)(x+1)$ is onto but not one-to-one.

Here I am concerned with the students' responses to whether f_i ($i = 1, ...5$) are one-to-one and onto. As far as the last part of Question 2.3 (request for an example of a function that is onto but not one-to-one) is concerned, the students almost unanimously produced $f(x) = x(x-1)(x+1)$, an example accidentally offered by the lecturer in the Question Clinic. As a result I consider this part of the data to be not valid. However a relevant observation on this part of the students' responses is that only a few offered justification for this choice of example. This justification was verbal (applying the definition for 'onto' and offering a counterexample for 'one-to-one') or graphical (resorting to the graph of f for drawing a conclusion). I elaborate these and other issues in the following.

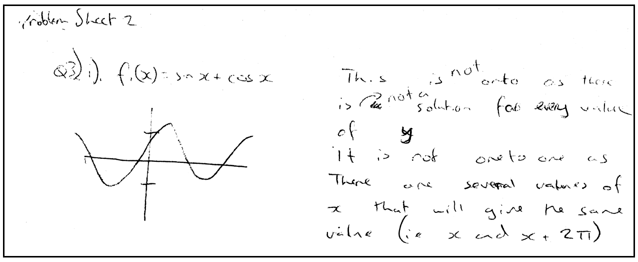
The major issue that emerged from the scrutiny of the students' responses to Question 2.3 regards an apparent lack in emphasis in the students' understanding of the concept of function as inextricably connected with all three of the following elements: relationship, domain and co-domain. A concern that emerges from this issue is the impact that this lack of emphasis may have on the students' study of properties such as onto and one-to-one. The format of Question 2.3, as appearing in the Problem Sheet, suggests that this lack in emphasis may be partly of curricular origin: the requirement that f_i are all defined from $\Re$ to $\Re$ appears on the first line. Then the five relationships follow. Is this format reinforcing what is known to be (e.g. Ferrini-Mundy and Graham 1991) the students' strong association (identification?) of the concept of function with an algebraic expression of this sort?

The distinctive elements in the students' responses that I wish to discuss here are:

A. Use of graphs to identify properties of the function: the students' responses varied from relying completely on the graph for mere identification of the property (relying on graphical representations for drawing inferences, but then not offering formal validation of their inferences, is a well-known characteristic of the students' reasoning at this stage - see, for example, Artigue and Viennot 1987) through to substantiating what the graph suggests with verbal explanations that included uses of B and C below. As an example, here is William's response to Q2.3(v):



Here William relies completely on the appearance of his graph for f_5: he grounds his claim that f_5 is not one-to-one on indicating two points of the graph, *a* and *b*, that appear to correspond to the same value on the *y*-axis. He then grounds his claim that f_5 is not onto on his graph-based observation that f_5 has 'maximum and minimum values in the *y*-axis'. The step towards a more formal grounding of these claims is not taken leaving William's reasoning vulnerable, for example, to possible inaccuracies of the graphical representation such as the ones in another student's, Luke's, graph for Q2.3(i):



In the literature I have quoted above the tension between the students' reliance on graphical representations and the need to formalise their insights is a well-researched area of learning difficulties at this level.

B. Use of the formal definitions of onto and one-to-one: the majority of the students' verbal statements were found to be confused, vague, often illogical and syntactically incorrect. I exemplify these later in the paper.

C. Use of properties specific to f_i (e.g. uniqueness of cubic root, uniqueness of solution of linear equation, boundedness of certain trigonometric functions): this usually resulted in not only correct but also justified answers. The students have discussed most of these properties at GCSE - A'level but resorting to them may impinge on another issue: what is the students' perception of what they are allowed to assume at this stage? If they are told that they are only allowed to ground their reasoning on already proved statements, to what degree their knowledge of these properties is established enough to allow this use (Nardi 1997)? As an example here is Wayne's response to Q2.3(ii), where he assumes the uniqueness of solution of a linear equation:

ii) $f_2(x) = 7x+3$
f_2 is a bijection, $\forall x \in R\ \exists f(x) \in R$ and also there is one and only one solution to the equation $y=7x+3$ $\forall y \in R$.

On the students' use of the formal definitions of onto and one-to-one. A large number of the students' verdicts on whether f_i are onto and one-to-one was correct. However these verdicts were rarely justified in sufficiently formal terms. To comment further on the students' responses, let us consider the expressions for f: $\Re \to \Re$:

E1: $\forall x \in \Re$, $\exists y \in \Re$, such that $f(x) = y$

E2: $\forall y \in \Re$, $\exists x \in \Re$, such that $f(x) = y$

E3: $\forall y \in \mathrm{Im}f$, there is exactly one $x \in \Re$, such that $f(x) = y$

E1 expresses that f is a function from $\Re$ to $\Re$. E2 that f is onto and E3 that it is one-to-one. The difference between E2 and E3 is a subtle one: in E2 there needs to be *at least* one x in the co-domain, in E3 *exactly* one in the image of f. Of course an understanding and use of E1-E3 depends heavily on the notion of $\Re$ as the Domain, the Co-domain and, tentatively, the Image of f. If f is a function, then every element of the domain must be assigned to an element of the co-domain. If f is onto, then for every element of the co-domain, there must be an element of the domain that is assigned to it. If f is onto, then the co-domain and image of f coincide. Therefore, to understand what it means for a function to be onto, one needs to understand the distinction between co-domain and image (a distinction that appears less problematic in the writing of students who resort to the graphs to discuss the properties of f_i).

Few students directly attempted formal expression such as E1 - E3. Most attempted a translation of their intuitions on f_i into verbal statements and it is mostly these statements that I am concerned with here. Amongst the few who engaged with quantified statements*, their formal mathematical writing appeared problematic. As Paola Iannone and I have written elsewhere about the students' attempts at translating their thought into formal mathematical writing (Iannone and Nardi, submitted), I omit further elaboration on this here and turn to a discussion of the students' verbal responses and, in particular, those that are problematic translations of the students' correct intuitions about f_i.

When engaging with justifying their claims that one of the f_i is not onto or one-to-one, the students use counterexamples (indicate x_1 and x_2 such that $f(x_1) = f(x_2)$ for one-to-one; indicate one y for which there is no x such that $f(x) = y$ for onto). Their approach is often determined by the use of a graphic calculator for identifying these values which are often estimates. These responses* appear to be a product of a different process to, for example, Luke's 0 and 2π (see his response earlier) who seems to enact an understanding of trigonometric functions as periodic. Moreover, as when relying on graphs (see A earlier), the student's responses are vulnerable to possible inaccuracies of the estimate.

Where the students appear less comfortable is with establishing an affirmative response, namely that one of the f_i *is* onto or one-to-one. Their resort to the formal definitions often is restricted to a mere citation of the definitions in the abstract*. Apart from citing definitions in the abstract, students seem to confuse E1 with E3 when they claim, for example, that f_2 is one-to-one because 'every value of *a* has only one corresponding value of *b* when $f(a) = b$' (Jo). And, for f_3: 'there is no $f(a)$ which has the same value as $f(a')$' without specifying the relationship between *a* and *a*'. It is noteworthy that Jo's arguments carry on similarly across Q2.3. Another, perhaps less clear-cut, fusion of E1 with E3 can be seen in another student's, Hazel's, response to Q2.3(iv): 'the function produces all possible R outcomes and they are all unique'. The first part of Hazel's argument is her reasoning on why the function is onto. The second part, where 'they' refers presumably to 'outcomes' suggests that different inputs produce different outcomes, therefore the function is one-to-one. However Hazel does not offer this clarification. If 'unique' refers to the uniqueness of the cubic root, then Hazel's response is problematic because the cubic roots in question are 'inputs' (a word she uses elsewhere in her writing), not 'outcomes'. Statements of similarly debatable precision - such as e^x is not onto 'as the graph does not extend infinitely into the negative' - are very frequent in the students' writing and reflect commendable but unsuccessful attempts at verbal explanation.

A dramatic and typical fusion of E1 and E2 can be seen in Louise's response to Q2.3(ii) and of E1 and E3 in her response to Q2.3(iii):

ii) $f_2(x) = 7x+3$

This is one to one as for every value of x there would be a corresponding y value.

Louise seems to recall in part (iii) that being a function entails a certain kind of full coverage; but she is not clear of which set (domain or co-domain). She even pursues a transformation of the co-domain so that f_3 'would be' a function.

Also (ii) is a fusion of all E1 to E3.

The ubiquity of this confusion in the students' responses indicates a matter of pedagogical urgency: E1 - E3 need to be unpacked for the students and made distinct; the concept of function needs to be discussed emphatically in terms of its domain, co-domain and image in order to counterbalance previously constructed concept images of a function as simply a relationship between numbers and in order to clarify that not all relationships are functions and that image and co-domain do not always coincide.

NOTE

The title of the project is *The First-Year Mathematics Undergraduate's Problematic Transition from Informal to Formal Mathematical Writing: Foci of Caution and Action for the Teacher of Mathematics at Undergraduate Level.* It is an Action Research Project (Elliot 1991). Phase 1 was conducted in 6 Cycles of Data Collection and Processing following the fortnightly submission of written work by students during a 12-week term. Within each 2-week cycle students attend lectures and problem sheets are handed out; students participate in Question Clinics, a forum of questions from the students to the lecturers; students submit written work on the problem sheet; students attend tutorials in group of six and discuss the now marked work with their tutor. Dr Iannone and I then engage in an analysis of the student's written work. We report details of the analytical procedure in (Nardi and Iannone 2000).

* Evidence for the claims made here that carry an * was presented at the conference but had to be omitted here due to limitations of space.

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