RESEARCHING STUDENTS' SYMBOL SENSE

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In 1994/5 the Royal Society and Joint Mathematical Society set up a joint working group on the teaching of algebra in schools and as a result published a Report "Teaching and Learning Algebra pre 19" (Royal Society, 1997). The report was the catalyst for my research for my Masters dissertation into high attainers' understanding of symbols. I have used classic errors and misconceptions to inform my research which involved students from years 10, 11, 12 and 13. In this paper I briefly explore some of the implications for teaching which have emerged from part of the research.

INTRODUCTION

As a practising teacher I am aware of the difficulties students have with algebra when they first enter the sixth form and how lack of fluency can impede their progress. The Royal Society Report (1997, pp 5) quotes Arcavi's (1994) article on *symbol sense* as, by implication, one exemplar, of the kind of algebra that they think we should be teaching. For my dissertation I researched students' symbol sense basing my own understanding on Arcavi's catalogue of behaviours which characterise symbol sense. This includes, among others, an understanding of the power of algebra, a feeling for when to abandon symbols in favour of other methods, an ability to manipulate and "read" symbolic expressions, the realisation of the constant need to check symbol meanings. Symbol sense is not just being able to manipulate symbols fluently. It "should become part of ourselves ready to be brought into action at almost the level of a reflex" (Arcavi, 1994, pp 32).

BACKGROUND, SAMPLE AND METHODOLOGY

On the premise that symbol sense is not easily acquired my case study only involved years 10, 11, 12 and 13. For ease of access my sample was students from the school in which I teach. The school is an Independent girls school in an affluent suburb of London. Mathematics GSCE results range between A* and C, with a very occasional Grade D. Similarly at A Level the grades obtained are usually between A and C.

The size of the sample was dictated by the fact that only nine students were studying A Level Mathematics in year 13, so I used one for the pilot and the remaining eight for the study. Additionally a random sample of eight students from each of year 12 and from the top set and top half of the second set in years 10 and 11 was used.

Each pupil was given a ten question questionnaire to complete in their free time composed of questions based on different aspects of symbol sense. Various scores were awarded for each response, a spreadsheet used to summarise and manipulate the data to obtain percentages and means. A sample size of only thirty two meant that statistical testing of the data was not viable. Following analysis of the data, two students from each year group were interviewed.

THE STUDY

In this paper I examine the teaching implications from three questions which used equation like statements.

Question

Look at the following

$$\frac{2x + 4}{x + 2} = 10 \qquad \text{(line 1)}$$

$$2x + 4 = 10(x + 2) \qquad \text{(line 2)}$$

$$2x + 4 = 10x + 20 \qquad \text{(line 3)}$$

$$4 = 8x + 20 \qquad \text{(line 4)}$$

$$-16 = 8x \qquad \text{(line 5)}$$

$$-2 = x \qquad \text{(line 6)}$$

Is this answer definitely true? possibly true ? never true?

State how you know.

If a student has symbol sense then (s)he “will defer the ‘invitation’ to start solving and instead try to ‘read’ meaning into the symbols” (Arcavi, 1994 p 27). The equation like statement in line 1 normally would be automatically manipulated by students in order to obtain a solution, so this question was included to see if students would halt their initial impulse to manipulate the symbols and "read" them first. Only two pupils, both in year 13, noticed that the numerator is double the denominator, so that the left hand side must equal two not ten. The other students’ responses to the questions was two-fold:

To Substitute

They substituted $x = -2$ into the equation. This gave a zero denominator which only two pupils, both in the sixth form, realised is not possible so that that the original equation like statement is never true. The majority of the pupils who substituted $x = -2$ ended up with 0/0 = 10. They concluded 0/0 = 0 not 10 so the answer $x = -2$ is never true.

To Rework the Equation

Many pupils reworked the equation, used the same method as on the questionnaire and concluded the solution $x = -2$ was definitely true. Because the algebra worked out 'nicely' it didn't occur to them that the solution could be anything other than definitely true.

The students' methods of checking in another question were similar.

Question

Micheala was asked to solve the equation $2y^2 = y$. Here is her solution:

$$2y^2 = y$$
$$2y = 1$$
$$y = \frac{1}{2}$$

Do you think her solution was

the complete correct solution? a partial correct solution? an incorrect solution?

When solving a problem, a student with symbol sense would realise the need to check symbol meanings and compare those meanings with the expected outcomes of the problem (Arcavi, 1994 p 31). Would students recognise that $2y^2 = y$ could have two solutions and manipulate the equation appropriately? A quarter of the sample stated that the solution was partially correct with the right explanation. Half of these were able to "read" the symbols and just stated the error, half reworked the equation correctly. Several pupils divided by y, just like Micheala, and seemed to be unaware that they might be dividing by zero. When interviewed, the year 13 students realised the danger in dividing by y but one of the year 12 students, when asked "What has Michaela done?", said:

61E: She has divided by y

I: What's the danger?

61E: When you divide by y……she left one in its place there…..she should have a plus or minus answer for something that's squared. Maybe she should have square rooted both sides instead of dividing by y. I can see what she's done……I probably would have done the same.

I: What's wrong with dividing by y?

61E: Is there anything wrong with dividing by y? I'd have said it was sound - you could divide by y….those are times they're not added together so who's preventing you from dividing by y?

The year 10 pupils were asked:

I: Is there any number you can't divide by?

10C: Zero - I don't know why

A similar lack of understanding was displayed by a year 11 pupil:

I: Do you know what 0/0 means?

11H: Is it infinity? No….I don't know

I: What about dividing by zero?

11H: You can't divide by zero can you? There is not a number so you can't get a number.

Dividing by zero is conceptually very difficult and although it is clear from the above that the students have met the idea, they have not understood it.

Others who reworked the algebra could see nothing wrong with the given solution, $y = ½$, concluding that it was completely correct. There were several different algebraic errors in reworking the solution to $2y^2 = y$ such as square rooting. At the interviews it transpired that the y^2 signalled 'square root' which in turn signalled two solutions, a positive and negative one and some students told me the two solutions were $\pm ½$.

Some pupils used a substitution method as in question six, also concluding that the solution was completely correct. Had the students checked for symbol meanings (Arcavi, 1994, pp 31) they would have been aware that ‘ y^2 ’ signalled **two** solutions.

Question

Some students were trying to solve a problem. Their solution was:

$$\frac{8y - 6}{4} = ½\,(4y - 3)$$

$$2y - 6 = 2y - \tfrac{3}{2}$$

$$-6 = -\tfrac{3}{2}$$

They knew their answer was silly but could not spot what they had done wrong. Can you?

Having symbol sense “is at the heart of what it means to be competent in algebra” (Arcavi, 1994 p32). Would students in this study be sufficiently competent to recognise the error in the question and manipulate the equation correctly? The question is similar to one set by Lee and Wheeler (1989) who found that students either cancelled part of the numerator with part of the denominator or solved the equation.

Only two out of the thirty two pupils in this study repeated the error in the question by incorrectly cancelling the fraction and just over a third were able to give the correct explanation. Of these, only four were able to "read" the symbols and give a two line written explanation, with nearly all the others reworking the equation by dividing the numerator correctly by 4. The remaining students also re-worked the algebra, usually, but not always, by multiplying both sides by 2 or 4. Those who multiplied had been taught to consider $8y - 6$ as a whole unit (interviews) , their teachers pre-empting the common error of students cancelling only part of the numerator with the denominator.

Responses by Number of those students who obtained an identity for Question 4

	Pre GCSE	Post GCSE
No Explanation of error in question 4	3	3
Incorrect Explanation of error in question 4	2	1

y = a solution	3	0

Several ended up with an identity such as 8y - 6 = 8y - 6 which there were unable to interpret (see table above).

Two year 12 students were asked by me (I):

I: What does 8*y* - 6 = 8*y* - 6 mean?

61G: That means one side equals the other side

I: What does that mean?

61G: The two sides are equal

61C: There is no solution

I: No solution. [to 61G] What do you think?

61G: y = 0

I: You think there is one solution?

61G: or like ….yes, one solution. *y* is the same amount to make both sides equal

I: Are you saying the same amount is one particular value?

61C: If you work out the equation 8y - 6 = 8y - 6 if you take the *y*'s over to one side you are going to eliminate the letter y so you haven't got any solutions, its been worked out the wrong way. You get nothing equals nothing, so there must be another way so you still have *y*.

61G: In this case *y* must equal nought to make it nothing

61C: I disagree, I don't think…..I dunno, you'd be able to find out if *y* equalled nought say if that was 7, 7*y* you could take the minus 7*y* over to there so you'd have 1*y*, then plus 6 over, so 1*y* = 0 and you would have a value for *y*, but there is no value for *y* here

I: So what do you think 8*y* - 6 = 8*y* - 6 actually means, what's our conclusion?

61G: There is no solution we've decided

61C: Yes

61G: Both sides of the equation equal each other

61C: There is no solution to *y*, there might be another way of doing the algebra -

61G: - that we've missed

61C comes across as being prepared to think through the implications of 61G's suggestions but is still very confused about the meaning of the identity. She seems to think that the equation **must** end up with some value for *y*, realises that *y* cannot be zero, but since she has no other means of interpreting the identity she concludes “there is no solution". “If the student, unable to see the abstract objects behind the symbols, is 'programmed' to regard a problem as solved only when an expression of the form '*x* = number' or '*x* > number' is obtained, then in a situation in which such an expression does not appear at all he or she must feel lost and helpless" (Sfard and Linchevski, 1994 p 222).

REFLECTIONS

This research has had an effect on some of my teaching practices. For example this term, when *estimating*, several students in year 9 ended up dividing by zero. I made the decision to take them through what happens as we divide by a smaller and smaller number. They didn't all like it, but it was a beginning. When teaching the sixth form I have made a point of not just cancelling out terms across an equation, but specifically stating, say, $m \neq 0$ so we can divide by m.

In all my classes now one of my questions is often "how can we check our answer?". My research and experience as a teacher has shown me that many students do not bother to check solutions. If they do, it is usually by substitution or reworking and if they get a different answer they don't know what to do next. Perhaps, with our more able students, we shouldn't shy away from teaching them about algebraic structure, albeit in an informal user-friendly way.

Nearly all the students interviewed, off tape, said they had never thought so much about mathematics. So far I have been unable to do anything about giving enough time to the students in lessons for reflection. The dual pressures of the syllabus and assessment are relentless.

As yet I have not had the opportunity in lessons to discuss what an identity such as $8y - 6 = 8y - 6$ means because the situation has not arisen. In order for students to learn about this and other conceptual points I may have to design my teaching to focus on these by provoking cognitive conflicts (Bell, 1993) within the curriculum.

Questions which don't "work out nicely" and other things that happen at the fringes of mathematics are often useful in testing students' conceptual understanding. Used carefully within our lessons they could be powerful instruments for developing high attainers' understanding of algebra and improving their symbol sense.

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