ADJUSTING TO THE NORMS OF MATHEMATICAL WRITING: SHORT STORIES ON THE JOURNEY FROM *CIPHER* TO *SYMBOL*

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To cipher is to 'express, show forth, make manifest by any outward signs, portray, delineate', to 'express by characters of any kind', to engage with a 'secret or disguised manner of writing, whether by characters arbitrarily invented or by an arbitrary use of letters or characters in other than their ordinary sense, by making single words stand for sentences or phrases, or by other conventional methods intelligible only to those possessing the key' ... (Oxford English Dictionary). Here we draw on a small study of the transition from informal (school) to formal (university) mathematical writing and discuss examples of Year 1 mathematics undergraduates' written work that illustrate their authors' variably successful but often endearing attempts at adjusting to the norms of mathematical writing.

As suggested by the absence of a reference to results, findings etc. in the above Abstract (a common practice is to include one), this is a preliminary attempt at disseminating a few observations on a set of currently collected data on the transition from informal (school) to formal (university) mathematical writing. In the following we introduce our project very briefly and then discuss examples of first-year mathematics undergraduates' written work. The focus will be on the students' attempts at adjusting to the norms of mathematical writing.

This project is funded by the Nuffield Foundation and, at least its initial phase, will last 3 months (October - December 2000). It is located within a series of projects that the first author has been involved in for several years (see Note 1) and its title is *The First-Year Mathematics Undergraduate's Problematic Transition from Informal to Formal Mathematical Writing:Foci of Caution and Action for the Teacher of Mathematics at Undergraduate Level*. It is an Action Research project and can be seen as a natural descendant of its predecessors (see Note 2).

The aims of the study are: identifying the major problematic aspects of the students' mathematical writing in their drafts submitted to tutors on a fortnightly basis; increasing awareness of the students' difficulties for the tutors at UEA's School of Mathematics; providing a set of foci of caution, action and possibly immediate reform of practice; and, setting foundations for a further larger-scale research project.

The study is carried out as a collaboration between the School of Education (where the first author is a Lecturer) and the School of Mathematics (where the second author teaches the first-year undergraduates) at UEA. The focus of the research, examining the students' *written* expression, has been identified as a worthy domain of investigation in Projects 1-3: these studies examined the students' development of mathematical reasoning in the wider context of both *oral* and *written* expression - the latter merits further elaboration and refinement and has also been highlighted by teachers of mathematics at university level as an aspect of the students' learning that calls for rather urgent pedagogical action (e.g. Nardi 1999).

This is a small, exploratory data-grounded study of the mathematical writing of the students in Year 1 (60 students in total, 16 in the second author's tutorial class). It will be conducted in 6 cycles of Data Collection and Processing following the fortnightly submission of written work by the students during a 12-week term. Each 2-week cycle consists of the following stages:

- Beginning of Week 1: Students attend lectures and problem sheets are handed out.
- Middle of Week 1: Students participate in a Question Clinic, a forum for questions from the students to the lecturers.
- End of Week 1: Students submit written work on aforementioned problem sheets.
- Beginning of Week 2: Students attend tutorials in groups of six and discuss the now marked work with their tutor.
- End of Week 2: Data Analysis Version 1, towards Data Analysis Version 2.

The second author, who is also a tutor and is responsible for collecting and marking the students' work, carries out an initial scrutiny of the students' scripts and composes Data Analysis Version 1: this consists of a Question/Student table where each student's responses to (a selection of) the problem sheet's questions are summarised and commented upon. The focus of her comments is quite open at the moment and covers a large ground of content and format issues. In an appendix to this table she produces rough frequency tables that reflect patterns in the students' writing and informal commentary by the tutors who teach the rest of the 60 students. Following a detailed discussion of Data Analysis Version 1, the first author produces Data Analysis Version 2, a question by question table where the major issues are summarised, characteristic examples of the students' work are referred to and links with current literature are made. A large part of these discussions revolve around the exchange of ideas and expertise: examples of this exchange include the communication of the second author's experiences as a tutor and a mathematician as well as her observations of the lectures and the Question Clinic, her consultation of other tutors and lecturers involved with teaching the students in Year 1; also her introduction to relevant findings from mathematics education research and educational research methodology.

Version 2 is then available to the other tutors for further informal commentary (we intend to introduce more formal strategies of evaluation in subsequent projects). An outcome of the discussion on Version 2 is what we are now starting to call Macro and Micro Points of Action and we refer briefly to those towards the end of this paper.

By the end of the 12th week, 6 sets of data and analytical accounts as described above will have been produced. We intend to organise a Departmental Day Workshop to disseminate and discuss our results and also cultivate opportunities for extending the project towards an implementation of our Action Points.

As a sample of the currently collected data we now present examples of the students' writing out of a proof by contradiction required in Q1.1, the first question in the Week 1 problem sheet.

Question 1.1: Write down a careful proof that $\sqrt{2} \notin \mathbf{Q}$ (do this by contradiction: assume that $\sqrt{2} = m/n$ with n, m having no common factors and see where you get after squaring and clearing fractions).

The lecturer's answer to Q1.1 in his notes to the tutors: Assume that $\sqrt{2}$ is a rational number m/n, written in lowest terms. That is, m and n have no factor in common. Then $m^2 = 2n^2$, so m^2 is an even number. Now the square of an odd number is odd, so if m^2 is even, m must be even - say $m = 2k$. Now we have $4k^2 = 2n^2$, so n^2 must be even. As before, this forces n to be even. Thus both m and n are even, which contradicts the assumption that they have no common factor. So we can conclude that $\sqrt{2}$ is not rational.

As the students were firmly instructed towards a construction of a proof by contradiction, all of the students engaged with an application of these instructions. The striking element in their responses is their diversity, in particular as far the *appropriation of formal mathematical language* as well as *the employment of previously 'established' mathematical results*. The constraints in space allow us to share here none but one of the examples shared at the conference presentation: Laura's intriguing 'proof' based on her intuition about 'infinite' simplification of fractions. The contradiction implicit in the first lines of her draft escapes her and, instead, she engages in a long winded, if encouragingly articulate (given the notorious resistance to prose by mathematicians (e.g. Burton and Morgan 2000, however in the context of research mathematics)),'regression' to arguing within the familiar setting of numbers.

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1) $\sqrt{2} = \frac{m}{n}$

$2 = \frac{m^2}{n^2}$

$2n^2 = m^2$

Any number multiplied by 2 becomos even.
∴ m^2 must be even.
If the square of a number is even then the number is even.
∴ m must be even and can be written as $2x$, where x is some interger.

So $2n^2 = (2x)^2$

$2n^2 = 4x^2$

$n^2 = 2x^2$

∴ For the same reason n^2 must be even.
∴ n must be even and can be written as $2y$, where y is some interger.

If $\sqrt{2} = \frac{m}{n}$

then $\sqrt{2} = \frac{2x}{2y} = \frac{x}{y}$

where $\frac{x}{y}$ is a more simple fraction than $\frac{m}{n}$

But this process can be repeated on $\frac{x}{y}$ to obtain an even simpler fraction, which can be made into an even simpler fraction, and so on forever.

However, we know a fraction cannot be simplified forever, and so the above statement is absurd. This leads to the conclusion that $\sqrt{2}$ cannot be written as a fraction.

Returning to the reference to *ciphering* from the Oxford Dictionary in our Abstract and to the ideas by Foucault and Hall that have influenced the analysis in Project 1 (as in Sierpinska 1994), these attempts are no more and no less than pleas for inclusion in the new to the students culture of Advanced Mathematics. Let us raise now issues relating to this desire to be mathematical by forwarding to four weeks later in this course and on another proof by contradiction:

Question 3.5: Suppose A is an n x n matrix which satisfies $A^2 = O$ (the n x n zero matrix). (i) Show that A is not invertible. (ii) Show that $I_n + A$ has inverse $I_n - A$. (iii) Give an example of a non-zero 2 x 2 matrix A with $A^2 = O$.

The lecturer's answer to Q3.5i in his notes to the tutors: If $BA = AB = I_n$ then $O = B^2 A^2 = BBAA = I_n$ contradiction. So A is not invertible.

The specific issue we wish to raise here is that of the choice of method and context in the students' proofs. In fact 'to be mathematical' is an aspiration that the students materialise with hesitation when it comes to adopting a mathematical *modus operandi*. For example, in Q3.5, again, proof by contradiction was desirable - but not explicitly requested - as was an argument within the context of matrix operations. Few student responses though matched the question setter's intentions. Instead of this neatly contextualised Method-Context (1: Contradiction - Matrix operations), the majority of students made a 'reductive' choice of another Method-Context (2: 'straight' deduction - Arithmetic of Determinants). Here is an example - Nicolas:

(5) A is an nxn matrix $A^2 = 0$
(i) $\det(A^2) = 0$
$\det(A) \cdot \det(A) = 0$
$\det(A) = 0 \Rightarrow A$ is not invertible

Context 2 is closer to their school mathematical knowledge and rings bells of familiarity. It is a safer choice. But does it provide the students with the benefits from doing Q3.5 intended by the question setter?

The question was intended as an exercise in matrix operations, where matrices are treated as objects, their multiplication is non-commutative and their inverse must be written/checked from both sides; where the identity matrix is decomposed as the product of a matrix and its inverse and associativity is a property that helps us reshape an expression with brackets. Reducing this argument from an argument about matrices to an argument about their determinants (themselves numbers where all the above properties have been used in a trivial manner by the students throughout their years of schooling; AND a method not yet taught/proved in lectures; from school they only 'know' the 3x3 case) is slightly missing the point of engaging with the question. As for choosing Method 2 at the expense of Method 1, this does not have the same grave repercussions as the choice of Context 2 at the expense of Context 1, even though it can be alarming that the efficiency of Method 1 eludes the students. Finally, if Method 2 had been formally introduced, the flavour of the above may have been slightly different but not substantially so: in fact it is quite natural for a learner to resort to the resources that are more familiar, that yield a sense of ease and confidence. It is perhaps more of a criticism of the question if it doesn't succeed in triggering the student's choices towards the more beneficial. The intentions of the question may have not been transparent enough for doing so and the students' use of Method 2 is just another case of *the confusion with what knowledge they are allowed to assume* theme that emerged in Project 1.

Implications for Teaching - Foci for Action. As we hope the above illustrate, our preoccupation with the students' writing is a very thin disguise for our interest in their thinking and learning processes. This is what makes this semiological exercise pedagogically relevant. What we aim is that our data and analysis indicate issues towards which the tutors' awareness perhaps ought to increase. Our micro/macro foci for further action derive from this analysis: a micro-focus would be a recommendation regarding a specific mathematical topic (e.g. discussing with the students the concept of function in terms of its domain and codomain to counterbalance their concept images of a function as simply a relationship between numbers). A macro-focus, for example, would be a recommendation, for example on the basis of evidence we have presented here, on the need to debate with the students the value of responding to Q3.5 in both above described ways towards a more transparent understanding of the question's intentions. The macro-foci are now beginning to build up to a proposal for a restructuring of the tutorials towards more interactive formats. We hope to be able to present this proposal in subsequent publications.

NOTES

1. **Project 1:** a doctorate (Nardi 1996) on the first-year undergraduates' learning difficulties in the encounter with the abstractions of advanced mathematics within a tutorial-based pedagogy

Project 2: a study of the tutors' responses to and interpretations of the above mentioned difficulties (e.g. Nardi 1999), and,

Project 3: UMTP, the *Undergraduate Mathematics Teaching Project* with Barbara Jaworski and Stephen Hegedus, a collaborative study between researchers and tutors on current conceptualisations of teaching as reflected in practice and their relations to mathematics as a discipline (e.g. Jaworski, Nardi and Hegedus 1999).

2. We tend to think of this study as Project 4, not only for its obvious thematic links with the previous projects but because it carries further the methodology of partnership and materialises what was an underlying intention in Projects 2 and 3: the involvement of the mathematician as a reflective practitioner and her engagement with Action Research.

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