THE CONCEPT OF SUPREMUM/INFIMUM OF A SET: A PROBLEMATIC OVERTURE TO THE CONCEPT OF LIMIT?

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In their first encounter with the subtle concepts of supremum/infimum, mathematics undergraduates often construct perceptions such as: a set contains its infimum; the Approximation Lemma, the second condition of the concept definition, is redundant; any number smaller than the supremum of a set, must necessarily be in the set. Furthermore the students are perplexed with the alternation of the terms 'greatest', 'least', 'upper' and 'lower' in the concept definition. Given the epistemological relevance of supremum/infimum to the notion of limit and that, in finding suprema and infima, set-theoretical perceptions become implicated as do strategies for manipulating algebraic inequalities, these concepts provide a rich pedagogical milieu. Here I exemplify the above with a characteristic learning episode.

In the last 20 years (since e.g. Tall and Vinner, 1981) research has placed the focus of the students' difficulties with the limiting process in their encounters with the crucial notion of limit (of a sequence or of a function). In fact, depending for example on course structure, encounters with the limiting process may precede the students' encounter with the concept of limit. This is the case in courses, for instance, where the concepts of supremum and infimum of a set are introduced before the formal definition of limit - as is the case for the course where the data discussed here were drawn from [1] and where in the second week of the first term the students are asked to work on questions such as

Given $0 < \theta < 1$ define $\Theta = \{\theta^n : n \in \aleph\}$

(i) Prove Θ has an infimum and that $\inf \Theta \geq 0$.

(ii) Show that $\inf \Theta = 0$ (assume $\inf \Theta > 0$ and obtain a contradiction).

In the following section I exemplify certain students' emerging concept images relating to the concepts of supremum and infimum of a set in the context of a characteristic learning episode drawn from the study described in [1]. In the closing section I conjecture about a potential relationship between these concept images and the students' imminent concept images of limit - the introduction of which follows closely the introduction of suprema and infima in this undergraduate course.

THE OVERWHELMING LINGUISTIC AND CONCEPTUAL COMPLEXITY OF THE NOTIONS OF SUP AND INF

This Episode takes place in the second week of the first term in Year 1 and is in the context of a weekly individual tutorial. The tutor and student Cornelia discuss questions from the second problem sheet on the Continuity and Differentiability unit, and, in particular, question CD2.6:

> For S, $T \subset \Re$ and $k \in \Re$ define
>
> $kS = \{ka\colon a \in S\}$
>
> $S+T = \{a+b\colon a \in S, b \in T\}$.
>
> If S and T have suprema prove
>
> i. that if $k<0$ then kS has an infimum and $\inf kS = k \sup S$.
>
> ii. $S+T$ has a supremum and $\sup (S+T) = \sup S + \sup T$.

In part i Cornelia says she showed that inf*S* exists but she could not prove that it is equal to *k*sup*S*; also that she showed that *k*sup*S* is a lower bound for *kS*. Therefore she could not prove that *k*sup*S* is the greatest lower bound of *kS*. The tutor suggests 'following your nose through the definitions': let $b > ks$, where $s = \sup S$, and prove b is not a lower bound for kS, namely prove that there exists an x in kS such that $x<b$. So, he then asks, what happens if we divide $b>ks$ by k?

C1: You've got b/k is greater than s in...

T1: No, remember k is negative.

C2: It's less than...

T2: Now follow your nose, what do you know about *[b/k]*?

C3: It's got to be the greatest...

T3: No, come on. sup*S* is the least upper bound for *S*. So what can you say for b/k if that's the least upper bound of *S*?

C4: It's hard work...

T4: Yeah, it is. Here is the least upper bound and here is a number smaller than it. What can you say about it?

C5: That must be in the set.

'No, no, not necessarily!' exclaims the tutor and explains: since *S* can be a 'dotty kind of set', all you can say is that b/k is not an upper bound of *S*. Therefore there exists an a in *S* with $a>b/k$. Cornelia then points out that multiplying through with k gives that $ak<b$ and ak is the x we were looking for. The tutor concludes that this is an example of what he means by 'following your nose through the definition', that is 'Apply the definition at each different stage'. In this case, he adds, among all the things one could say about b/k, the useful observation is that b/k is not an upper bound of *S*. He then concludes: she could have picked up the right thing to say if she had focussed on her goal to prove that b cannot be a lower bound for *kS*.

The Tutor's Three No's to Cornelia. Cornelia has had difficulties and finally gave up proving the second property for the infimum of *kS*. During the tutor's closed questioning presentation of the proof, her three responses are successively met with his disapproval. The tutor's three No's to Cornelia signify three misunderstandings, on her part, that presumably have constituted part of the stumbling block that led to her giving up on the proof in the first place.

- First (C1), Cornelia ignores that k has been given as negative and, when dividing $b>ks$ by k, replies $b/k>s$, instead of $b/k<s$. This is a typical algebraic mistake, that at university level is usually attributed to carelessness, which Cornelia instantly corrects once her attention is drawn by the tutor to the sign of k.

- Secondly (C3), when asked what is s (that in the proof denotes sup$S$) she starts her response with 'the greatest'. She is interrupted by an impatient tutor who reminds her that s is the least upper bound for S. As in a number of other Episodes most students appear in a linguistic unease (in parallel with their difficulty with the new concept) with the terminology commonly used for supremum and infimum, that is 'least upper' and 'greatest lower' bound. Similarly to a student in another Episode, who at least three times corrects himself interchanging 'least' with 'greatest' and vice versa, Cornelia, if given the chance, might as well have done so. Coming to terms with a new concept and its complex terminology is a demanding task that novices frequently find hard to carry out. I also note here that the alternation of the terms 'greatest' and 'least' in CD2.6 is even more cognitively demanding because of the reversal due to the negative sign of k.

- The two incidents cited above psychologically build up to Cornelia's resignation, signified by C4 and to her subsequent third and essential flawed response to the tutor's closed questions. When asked about what can be said about b/k, a number shown so far to be smaller than the supremum of the set, Cornelia deduces that it 'must be in the set'. In her response there is evidence of a conception regarding the supremum of a set according to which anything less than the supremum is necessarily contained in the set. This is true for intervals like (a,b) where b is the supremum and anything close to b but less than b belongs to the interval. The tutor notes that the set in their proof can be a 'dotty' one, thus breaking up her *continuous and dense* concept image.

Also her response is an illustration of a major difficulty of mathematical reasoning encountered by students: confronted with an inequality such as $b/k<s$, the available next-steps (observations that will lead to the next step of the proof) are numerous. Whether the decision made by the student is a fruitful one relies on how well she has realised the existence of the various options and on how she will associate the data with the goal of the proof. In this case, Cornelia did not seem to have adequate control of her options even though her response seems to indicate that she follows the tutor's train of thought. In this sense, the tutor's suggestion 'following your nose through the definitions' is a recommendation the meaning of which is not as obvious to the novice as to the expert mathematician. The tutor's metaphor however is

insightful in another sense: it captures the detective skills required in most mathematical activities, namely that, given the clues, one is able to make a choice that will turn out to be effective. Cornelia's response in Extract 6.7 exemplifies the cognitive behaviour of a novice who has not mastered these skills yet.

A Note on an Ostensibly Happy-Ending Story. Cornelia, after a succession of flawed responses that have been dismissed and modified by the tutor, completes his last sentence correctly, leaving thus a last impression of successful fulfilment of the tutor's task (which is to help Cornelia understand the proof for CD2.6i). It is a methodological constraint of the study from where these data have been drawn that no further evidence is provided with regard to whether Cornelia's last utterance, $ak<b$, is merely a correct calculation, an automatic algebraic reaction to the tutor's $a>b/k$; or whether it is a meaningful contribution to his proof and one which shows that her flawed initial conceptualisation of the proof has been reformed. As an observer, my impression is that, through the end of the Episode, Cornelia remains overwhelmed by the complexity of the definitions of supremum and infimum and uncertain of her own understanding.

DISCUSSION

In the above, evidence was given of difficulties embedded in the notion of supremum and infimum of a set, and in particular of the notion of infimum as the *greatest* lower bound of a set. From algebraic mistakes in the handling of inequalities to the confused use of the new terminology, additionally perplexed by the alternation of the terms 'greatest', 'least', 'upper' and 'lower', these concepts emerge as essentially problematic. A particular conception revealed in the Episode was the student's belief that a number smaller than the supremum of a set, must necessarily be in the set. At the metamathematical level, the student seemed to be in hardship with confronting the multiplicity of options in the course of a proof and with co-ordinating a variety of information in order to pick an effective option. In other characteristic Episodes, not cited here due to limitations of space, the students are additionally perplexed by the Approximation Lemma, the second condition in the definition of supremum and infimum of a set; also they seem to claim that a set must contain its infimum.

A conjecture that emerges from the data exemplified above is that some of the widely known difficulties of the students with limits can be traced in their thinking before the formal introduction to limits, in this case in the context of sup and inf. Each one of the above difficulties can be seen to be about an aspect of the limiting process (Nardi in preparation). To illustrate the epistemological relevance of the concept of inf and sup to the notion of limit, consider, for example, the question cited here in the introduction: in this, the infimum of set Θ is the limit of the sequence of its elements θ^r, where $0 < \theta < 1$. This epistemological kinship may imply a cognitive association too. My claim is that the evidence here provides a certain (modest) support for this link.

Consider, for instance, the students' perception of a supremum of a set as an element of the set. This appears to be analogous to what Tall terms *generic limit property*: all the properties of the terms of the sequence also hold for its limit. Note here that Leibniz, as well as Cauchy, believed that the limiting function of a family of continuous functions must be continuous. As a result students believe lim0.999....<1 even when they can prove that $\sum 9/10^n = 1$: 1 is not of the form $9/10^n$, so it cannot be the limit of $\sum 9/10^n$. And, as repeatedly reported in various studies, students in this case also claim that $\sum 9/10^n$ tends to 0.999..., but has the limit of 1.

Moreover, underlying the concepts of supremum/infimum and limit is a notion of infinitesimal proximity, represented in the former by the Approximation Lemma (and, linguistically, by the use of the terms 'least' and 'greatest') and in the latter by the choice of ε. Along with the fact that both cases require intensive manipulation of inequalities, this underlying parallel also points at the link suggested above. It is hoped that future work will substantiate this link even further.

NOTES

1. The study (Nardi, 1996), where all the data presented in this article are drawn from, is the author's doctorate. It aimed at the identification and exploration of the difficulties in the novice mathematician's encounter with mathematical abstraction. For this purpose twenty first-year mathematics undergraduates were observed in their weekly tutorials in four Oxford Colleges during the first two 8-week terms of Year 1. Tutorials were tape-recorded and fieldnotes kept during observation. The students were also interviewed at the end of each term of observation. A qualitative analytical framework (Miles and Huberman, 1984) consisting of cognitive (e.g. as in Tall, 1991) and sociocultural (e.g. as in Sierpinska, 1994) theories on advanced mathematical learning was applied on sets of Paradigmatical Episodes extracted from the transcripts of the tutorials and the interviews within five topics in pure mathematics (*Foundational Analysis*, *Calculus*, *Topology*, *Linear Algebra* and *Group Theory*). This topical analysis was followed by a cross-topical synthesis of themes that were found to characterise the novices' cognition. The thesis can be browsed at http://www.uea.ac.uk/~m011.

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BIOGRAPHICAL DETAILS

Elena Nardi is lecturer in mathematics education at the School of Education, University of East Anglia. Her research is primarily in the area of learning and teaching of mathematics at upper secondary and tertiary level. Before moving to UEA she was a postdoctoral researcher at the Universities of Warwick and Oxford, where she also conducted her doctoral research. Her teaching at UEA includes contributions on Mathematics Education (PGCE SY) and Educational Research Methodology to a number of courses. She is also co-ordinating the School's EdD programme.