SOLVING A NON-ROUTINE PROBLEM: what helps, what hinders?

Paul Blanc
Saint Martin's College.

This paper reports on research in progress and in particular discusses the solutions of two sixth form students to part of a non-routine problem. Their solutions appear to show potential insights and possible pitfalls due to the specific form of representation adopted in recording results. The precise way that representations are generated as part of each student's text and how this text is laid out on the page are claimed to have a significant influence on the progress of their solution. The use of video recording as an additional methodological tool enabled the researcher to identify and confirm such effects.

1 Introduction and background.

My interest is how students tackle problems where generalisations are sought. What might lead them to attend to the problems in such a way that they are successful in obtaining a generalisation? How do they draw on their activity on paper in the generalisationprocess? I investigate this by examining through video recordings how students tackle problems which involve (non-routine) problem-solving activities such as those commonly found in GCSE coursework and more recently timed examination assessment of attainment target Ma 1 (DtE 95). I also note that making sense of non-routine problems forms part of the National Numeracy Strategy (DtEE 99).

In previous research (Blanc & Sutherland 96) we found that students exhibited two types of behaviour partially as a consequence of a routinised approach:

a) Students got stuck in certain types of representation (e.g. tables or algebra) failing to relate what they have been doing or creating to the results/formulas that they were seeking to conjecture/find;

b) Students interacted very differently with their text. The characteristic difference being that some moved flexibly around their solution text altering it in an iterative manner, whilst others worked down the page in a fixed linear way.

Follow-up interviews revealed significant differences in how the students had solved the problem to the interpretations obtained from the script alone.

2 Methodology

The aims of the current study are: to characterise sixth form students' approaches to a limited number of pure mathematics investigations, to identify and analyse the effects of different approaches to these problems and to identify factors which influence students' problem solving behaviours in these contexts. The subjects are Lower Sixth Formers (age 16/17) who have recently completed their GCSE course. Data collection is via: a) background information questionnaire;" b) individual problem solving sessions (with video recording of the student's solution generation); c) interview; d) micro genetic analysis of script and a full account of each solution from all data sources. The video recording is an additional instrument of data collection and analysis used with the intention of gaining greater validity of interpretation.

3 The problem.

In seeking to offer a challenging problem with characteristics of a GCSE examination/coursework tasks my recent study used a problem which I shall refer to as Football Scores.

Football Scores
i) A football match finishes with a final score of 3 – 1
a) How many possible half-time scores are there?
b) Generalise this result for any final score.
ii) If the final score of a match is 3- 1, one route to this final score is
$0 - 0 \rightarrow 1 - 0 \rightarrow 2 - 0 \rightarrow 2 - 1 \rightarrow 3 - 1$
a) How many routes are there to a final score of 3-1?
b) How many routes are there to any final score?

This problem was selected following extensive piloting of real examination problems offered at GCSE since it both engaged and challenged students. Firstly I would like to offer some characteristics of the problem which I would use to compare and contrast it with others of a similar nature.

It is a non-routine problem. No student (from more than 75 cases) knew the answer immediately or a technique which gave rise to an immediate general solution. A list of further characteristics follows. 1) The statement of the problem is short. 2) An immediate strategy might be to try specific cases. 3) Answers for" various specific cases can be found quite quickly. 4) Some care or systematic organisation is required not to miss some outcomes. 5) Results from the individual cases can be accumulated into lists or tables. 6) The problem can be subdivided into particular subsets of cases. 7) There are general rules that can be found. 8) These rules can be expressed algebraically. 9) There are two independent variables.

Solution to part i) : A score of a - b has (a+ 1)(b+ 1) half time scores.

4 Presenting two students with contrasting approaches.

I will now explore two contrasting attempts at part i) of the Football scores problem. A fuller picture of each student's work together with a broader general analysis will appear in later articles.

4.1 Claire's solution.

I begin with the very first piece of text produced by the student who I shall call Claire.

Extract 1

Football Scores

3-1 full time
half time scores...
3-1 2-1 1-1 0-0
3-0 2-0 1-0

7 half time scores

I would argue that the potential layout used here is advantageous but Claire omits the score 0-1. I note that from video analysis that the actual construction of this representation was as columns from right to left, i.e. 3-1 then 3-0 then 2-1 then 2-0 etc. I also note that solution for a final score of 3-1 was later corrected to 8. Claire then moves on to create a two column table of results directly underneath the above.

Table 1

Full time Score	Number of half time Scores
3-1	7
0-0	1
0-1	2
1-1	4
2-1	6

This table was constructed from information created on a separate sheet of paper (headed "working out") where for a each full time score Claire wrote out the possible half time scores to the right. An extract from this page appears below.

Extract 2

• 2-1 = 0-0, 0-1, • 1-0, 1-1, 2-1, 2-0
• 1-0 = 0-0, 1-0

One horizontal line is used to list the possible half time scores for each full time score. I will use an example to clarify the way Claire worked.

a) Claire writes down 4 the number of half time scores for 1-1 in the second column on her first page (table 1, row 4). She then writes 2-1 in the left column (table 1, row 5).
b) Claire moves to her working out page and writes down the possible half time scores for 2-1 (extract 2, line 1).
c) Claire returns to her first page and writes 6, the result she has obtained in the right hand column (table 1, row 5).

Every time Claire wrote a new full time score on her table she wrote out the half time scores in one horizontal line on her "working out" page, then she counted the number of half times she had found and wrote this total into her table. In this manner she separated out her generation of examples from the table which contained the results derived from the examples. Identifying this process of production and the micro genetic analysis of particular cases was possible due to the video record of the student's solution generation. Her later examples were organised using single variable manipulation, fixing one score and varying the other e.g. 1-0, 2-0, 3-0, 4-0, this led to results for the one score fixed at 0, 1 cmd 2.

Further evidence of her separation of activity can be seen from some adamant replies in the interview transcript:

PB But did you go back to the examples at all to try to see whether you could spot the way that they worked, if you like, by looking at the examples, or did you workfrom the table?

C *No, Ijust workedfrom the table"*

4.2 Dave's solution

I will now look at the solution of the student who I will call Dave. Here the specific layout of a single example gives an immediate visual clue.

Extract 3

A: i- 0-0 0-1
1-0 1-1
2-0 2-1
3-0 3-1

ii. Full time score X-Y.
Possible half time scores (X+1)(Y+1).

He constructed this text by writing out the left-hand column and then the right. His solution makes strong use of visual appeal of his first example.

P B And did you ... was there anything that you looked at in say - in the first question - that helped you see that it would be one number times another?

D *Yes, because it's ... the arrangement. It's hard to explain it. There is - like I've got two columns of four and it's four scores for the home and two for the away team.*

Dave then proceeds to produce anotht::r example as follows.

Extract 4

eg f/t : 4-3.

h/t possibilities

0-0	0-1	0-2	0-3
1-0	1-1	1-2	1-3
2-0	2-1	2-2	2-3
3-0	3-1	3-2	3-3
4-0	4-1	4-2	4-3

5 possible scores for home team - 0, 1, 2, 3, 4.
4 possible scores for away team - 0, 1, 2, 3.
5 x 4 = 20 possibilities.
$(x+1)(y+1)$

Again the construction of the tabular layout of scores was in complete columns from left to right. When questioned as to the reasons for the example he made the following response.

D *Well, I saw that straightaway, and did the example to prove it.*

5 Discussion

It is clear that Dave focuses on *visual* cues and through this sees the structure of a generic example in what might not usually be seen as problem where graphical representations can help. (I.e. this is this is not a traditional figurative counting problem such as Diagonals of a Polygon (see Blanc and Sutherland 1995). No other student in this study actually produced this specific representation as a first example.

The use of the term "prove" here needs further elaboration. Whilst his use of the term might be regarded as dubious what he writes is how to work out a specific but nonetheless generic example, since all other cases follow the same reasoning. The only thing which might be missing is the generalised form of the argument. It is interesting is that the student decided to write out another example. My hypothesis from the data sources is that the student's attention is focussed on communicating his findings.

In the case of Claire the swift move to separate out the generation of examples from the generation of any rules leads to the table as a separator of activity (Blanc & Sutherland 96). I find no evidence of attention to visual stimuli, yet the individual strategies she adopts, organised recording of results in a table, breaking the problem down by specialising (Mason et al 85) and single variable manipulation are sound and those that might be encouraged in a school classroom, either by teachers, examination boards, text books and examination study guides, and posters.

Claire works in an iterative manner. In previous studies, such working has been linked with engagement with the task and progress towards a solution. Although she does adapt and correct her different pages to correct omissions, she did not gain *insight,* as her horizontal listings did not offer the richness of the columns and rows. I believe that this is in part due to the two variable nature of the problem.

Rowland and Bills' (1999) discuss reasoning from examples to a general rule stressing the difference between structural and empirical arguments. *Empirical* arguments consider the form of the results alone, e.g. from a table of results (by inspection or by examining differences). *Structural* arguments look at underlying meanings, structures or procedures, e.g. from attending to the manner in specific examples have been constructed or deconstructed. They discuss how a *generic* example might be convincing or accessible (or both) and whether generic examples are convincing per se (for certain groups of students) when compared with formal proofs. One aspect of my research is to consider what might make the difference between empirical and structural reasoning. In the cases above I claim that Claire is adopting an empirical approach whilst Dave's argument is structural. Dave offers a general solution and a generic example that pays attention to the structure of his solution. The way in which the students present and re-present their solutions would appear to have a significant influence on their progress and possible argument forms for this particular problem. In particular, the separation of activity in Claire's solution confirmed through the use of video micro-genetic analysis would appear to lead her towards an empirical approach.

<u>References</u>

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