THE ROLE OF THE SEMIOTIC REPRESENTATIONS IN THE LEARNING OF MATHEMATICS

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Abstract

We argue that mathematics visualisation and the construction of concepts require the coordination of different representations that belong to different semiotic systems of representations. Unsuitable construction of a concept will impede the acquisition of more abstract conceptualisation. In this context, we will discuss about cognitive obstacles which some teachers and students have when facing mathematical problems related to precalculus and calculus topics. Mathematics visualisation in this context deals with an holistic vision that articulates several representations in a problem solving situation.

Introduction

From an historical point of view we can find that in the construction of mathematics visual ideas played a strong role in the development of concepts. But, the contradictions that emerged in the early Greek philosophy pointed out by the school of Parrnenides (e.g. paradoxes of Zenon) produced a change in the construction of a deductive science. Since the time of the Greeks we have been constructing a mathematical building where figures are not allowed to be part of the mathematical reasoning. Szabo (1960, p. 40) states that: *"Hipocrates of Quios, in his Quadratura Lunularum, had demonstrated inequalities that he could obviously do by simple illustration. This observation shows that Hipocrates did not trusted in simply visualisation ...* ".

In this process of formalisation the intuitive ideas were hidden by the formality of the mathematical presentation and the different representations that gave life to a concept were lost. Since then the primacy of the symbolic (algebraic) form was favoured by the mathematicians. In the context of learning mathematics, should we follow this trend? will it be enough for the students to learn mathematical concepts from a formal presentation?

There is a paradox in the construction of abstract concepts, to what extent do we need the semiotic representations to construct the abstract concept? Mathematics visualisation has appeared in recent literature as one of the fundamental aspects in understanding students' construction of mathematical concepts. We would like to discuss the need for a visual approach when teaching mathematical concepts.

Mathematics visualisation and semiotic system of representations

The emphasis teachers show in their algorithmic-algebraic approach when teaching mathematical concepts has been influenced by mathematics textbooks. Then they are promoting the avoidance of visual considerations when solving a problem.

Eisenberg & Dreyfus (1990, p. 25), have shown that: *"many students are reluctant to accept the benefits of visualising mathematical concepts".* They state that: " ... *thinking visually makes higher cognitive demands than thinking algorithmically",* and that: *"nonvisual frameworks are used to communicate mathematical ideas. This custom is based on the belief of many mathematicians, teachers, and students that mathematics is no visual.* " (p. 30). In this context they state that: *"Indeed, with respect to problem nine* [see below] *one typical calculus teacher (who also happens to have authored a calculus textbook) wrote:*

f'(-a) =(f(-a))' = (-f(a))' = -f'(a). **As the answer to the problem:**
"Given: f is a differentiable function such that f(-x) = -f(x). Then, for any given a
A) f'(-a) = -f'(-a)   B) f'(-a) = f'(a)   C) f'(-a) = -f'(a)
D) None of the above

How could we know the answer is correct? Can we doubt about the answer given by a mathematics teacher? Do we have different approaches to verify the answer? What happened with a graphic representation of a particular situation, can we have a second thought?

f(x) = x³

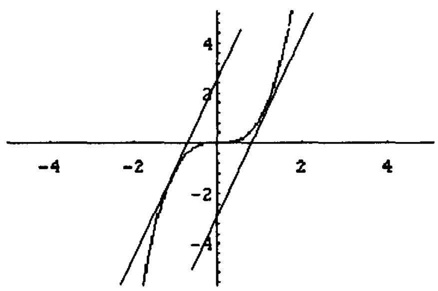


Figure 1

This is not a trivial example (see Figure 1); Indeed it is a non example! This is what we call a mathematical visualisation of the problem. With this approach we wanted to stress the importance of the complementary nature of the representations of a mathematical concept. Related to this, we can differentiate between an exercise and a problem, pointing out: *A non-routine problem is one that when reading it, an algebraic process does not come to our mind; then, we need an interpretation of the problem and surely the different representations we associate with it are going to play a principal role in the solution of the problem* ".

Curriculum and semiotic systems of representations

Related to the curriculum in middle school, some researchers are pointing out the need to break the anti-illustrative tendency in the teaching of mathematics to promote the learning of mathematics. Ben-Chaim et al. (1989, p.55), state that: *"the nature of visual representations, permit the vast majority of the students of that level to understand an informal presentation of a* *mathematical deductive proof, while an algebraic treatment could be far away from his comprehension"*

Let's see an example given by Selden et a1. (1989, 1994), where they have shown that students who finished a Calculus course could not solve some non-routine problems, they state: *"Not one student got an entire problem correct. Most couldn't do anything ... ".* One of those problems is presented in the Figure 2 (left).

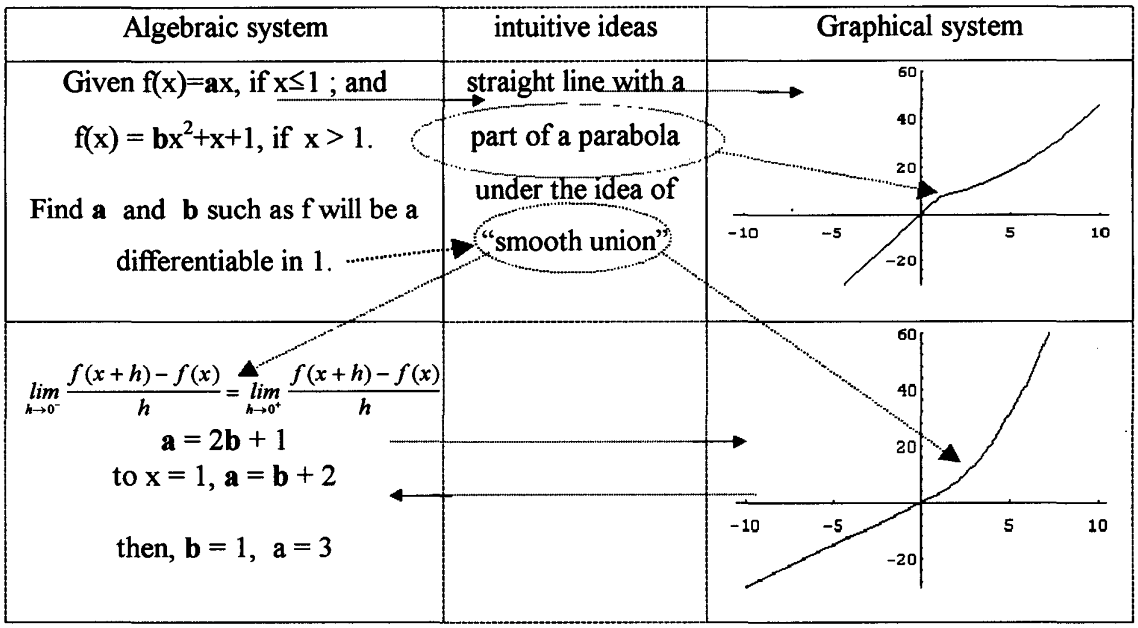


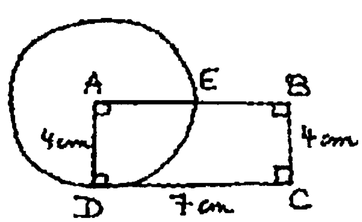
Figure 2

Did the students need a graphical approach to solve the problem? How can they relay the graphical interpretation of the problem to guide their algebraic approach? Do intuitive ideas play any role in this context? We think they do.

Is there a danger in the learning of mathematics when using figural representations that appeal to intuitive ideas?

It is important to point out that not everything is good when promoting the use of visual considerations. Do cases exist where an unsuitable interpretation promotes the error?

Let us look at an example given by Pluvinage (1998, p. 126): *"The following example, from Bernard Blochs, has been experimented on March and September of 1997 ...*



In this drawing done by hand, we draw:
- a rectangle ABeD
- a circle with centre A and passing by D
The circle cuts the segment AB at the point E.
What is the length of the segment EB ? Justify your answer: ...

The results obtained by 167 *school children of eleven years old* [College in France] *are:*

3 cm : related to a mathematical model	*18 %*
1.9 or 2 cm: related to a measurement with a ruler	*28 %*
3.5 or 4 cm: answer influenced by visual perception	*52 %*
Other answers	*2 %*

The percentage of students who followed an incorrect reasoning was very high!. We can see that in this case the iconic images can playa different role. Then ...

Theoretical aspects related to the semiotic systems of representation

Research on mathematics visualisation and the role of mental images have shown the importance of mathematics representations to the learning of mathematics (see for ex., Duval, 1993). We can observe in the work of Duval (p. 40), his characterisation of a register in a semiotic system of representation: *"a semiotic system could be a register of representation, if it permits three cognitive activities related to the semiosis:*

1) Identification of the representation in the system ...

2) Treatment of the representation that is the transformation of the representation in the same register where it was formed ...

3) Conversion of the representation that is the transformation of the representation in another register where it conserves the totality or part of the meaning of the initial representation ...

In this trend we did several experimental studies (see e.g. Hitt, 1994, 1998a & 1998b) related to the knowledge of teachers of mathematics (High School level) related to the concept of function. We found different kinds of obstacles, for example, given some graphs, the teachers were asked to draw a container related to the physical phenomenon (filling a container with some liquid); in one question the teacher's answer was:

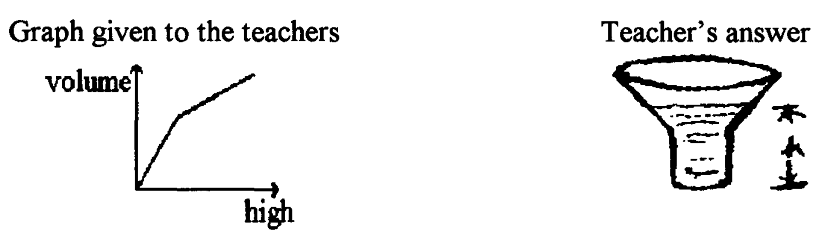


Figure 3

The primacy of a global intuition on the analytic thought did not permit the teacher to undertake a suitable reasoning to give a correct answer. In another questionnaire (in the same study) the same teacher gave the following answer to the question: Given a container like that in Figure 4, make a graph showing how the area of the liquid at the top varies according to the high of the liquid.

If we look at the answer related to the first question of drawing a container, we could say that this teacher is unable to undertake an analytical reasoning. But, analysing the other question where the teacher is asked to draw a graph, we see that he could develop an analytical reasoning of the situation to give a correct answer. We see that the articulati'on between representations is following ways that depend on how, as in this case, the teacher is recalling some knowledge he possesses. In the first question, his intuition played a greater role than his analytic reasoning.

The problem seems to be much more difficult than people usually think. When solving a problem, at what moment do we need to transform one representation into another?

We claim that we need several representations that complement each other to understand and to solve a problem. But, if we are augmenting the representations of a concept, are we making its acquisition more difficult?

Transcription of the teacher's work

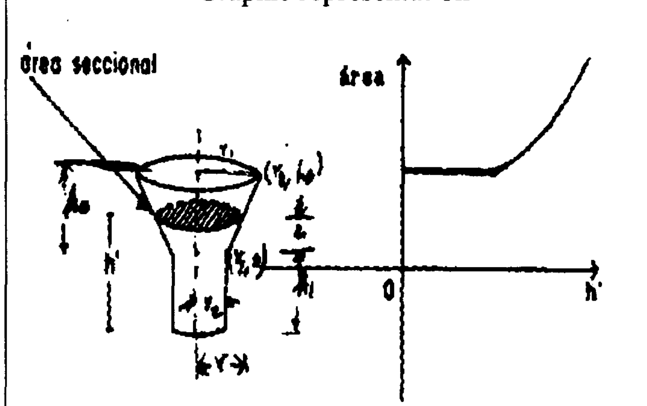
Teacher's algebraic process	Graphic representation
$h=\frac{h_0-0}{r_1-r_2}=\frac{h_0}{r_1-r_2}(r-r_2)$ $h=m(r-r_2)=mr-mr_2$ $r=\frac{h+mr_2}{m}=\frac{1}{m}h+r_2$ $Area=S=\pi r^2=\pi(\frac{1}{m}h+r_2)^2, h\in[0,h_0]$ $h'=h_1+h=h_1+mr-mr_2$ $r=\frac{h'-h_1+mr_2}{m}=\frac{1}{m}h'+\left(\frac{mr_2-h_1}{m}\right)$	

Figure 4

These ideas are summarised in the following diagram (see Hitt, 1998a), which is similar to that presented by Duval (1993); but, we included the idea of cognitive obstacle in there (Figure 5).

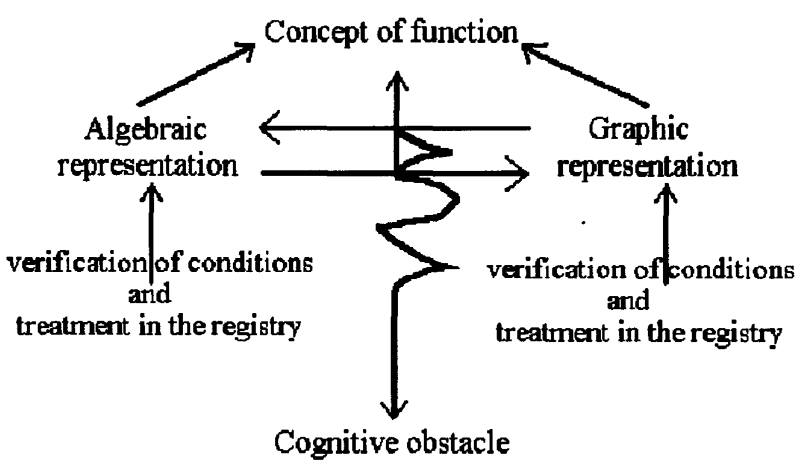


Figure 5. Mathematical and epistemological problems

Discussion

We have shown that in recent research we can observe that high school students and even some teachers are not able to articulate several systems of representations related to concepts working at that level. The teachers tend to fix on some strategies focusing on algebraic methods, and because of that, the student will experience some problems when they need to do a transformation from one system of representation to another, like from the graphic to the algebraic system (see for example, Ruthven, 1990). Then, the students learn to transform coherently the representations in the same semiotic system, that of algebraic system.

It is important to promote the use of several systems of representations and the reflective use of new technology that allows the student to give a meaning to the mathematical notions he/she is learning. By this process: *the construction of a concept is related to the coordination, without contradictions, of the different semiotic system of representations of the concept.*

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