Advanced Mathematical Thinking Working Group: Summary of Birmingham and Leeds meetings

Convenor: Stephen Hegedus, University of Oxford
amt@soton.ac.uk (or heg@ermine.ox.ac.uk)

Meeting held at Birmingham University, June 1998

The last session highlighted an interest in the idea of the procept *in elementary mathematical thinking and its possible use in transfer to advanced mathematical thinking (AMT). This session set out to analyse the paper by Gray & Tall* (1994), *to enlighten the group's understanding of the idea and its consequences. The group offered mixed reactions towards the paper and advanced mathematical thinking as a genre. This paper highlights the main themes of the paper, the reactions of the group and future ideas for discussion.*

The procept can only be described by first observing a mathematical thought process as a process and a concept and then allowing the perceived dichotomy between the two to actually converge into a flexible cognitive operation. Gray & Tall (1994) consider the duality between a mathematical process and concept and address the ambiguity in the notation as allowing a successful thinker the flexibility to operate between the process and the concept to be cognitively manipulated.

Gray & Tall hypothesise that children who succeed in mathematical problemsolving are doing something qualitatively different mathematically. They define the *process* to mean 'the cognitive representation of a mathematical operation' and the *procedure* as a 'specific algorithm for implementing a process'. More importantly, flexible thinking using conceptual knowledge is likely to be different to thought processes manipulating inflexible procedures. It is the operation of observing mathematical processes as mental objects that can be manipulated which allows the process to be conceived as an a concept. Other authors have observed such cognition (Dubinsky, 1991; Sfard 1989,1991) but not dualistically. The perceived problem of the existence of an object and a process existing at the same time is alleviated by the authors by the examination of the role of the symbol. The explain how mathematicians use the same notation to represent the process and the product of that process. For example a/b and *a divided by b* are synonyms (re Thurston, 1990).

The authors coin the portmanteau 'procept' for the operation of viewing and utilising both in a dualistic and a mutual sense in an integrated piece of cognitive development. For example, the procept 6 may be flexibly decomposed and recomposed in several ways: $6 = 2 + 4 = 2 \times 3 = 8 - 2 = 6$. It is the ambiguity of symbolism which allows flexibly duality of process and concept and this cannot be utilised if distinctions between the two are continually made. Proceptual thinking is referred to as the combination of conceptual and procedural thinking, and it is the symbol, itself, which expresses conceptual and procedural links, processes and the product of processes.

The paper outlines studies doe by Gray (1991) interviewing a cross-section of children aged 7-12 from two mixed-ability schools. Three addition and three subtraction problems were used and results showed a significant difference between the groups with respect to the procedural methods used. The low-ability groups used count -all and count-on far more than the high-ability groups who were using procedural (counting) methods and derived facts more extensively. Similar results were obtained with the subtraction-problems and the use of derived (subtraction) facts.

Gray & Tall (1994) provide empirical evidence which supports the idea that there is a proceptual divide in elementary the thinking of children. This arises because of significant differences in the way less-able thinks flexibly use the process and the concepts they have. The less-able do not simply learn techniques more slowly but use different techniques. They rarely used derived facts and this produces a high cognitive load which prevents the successful completion of problems and solving problems at a higher-level by only co-ordinating sequential processes.

For the more able, proceptual thinker:

> [C]ounting, addition, and multiplication are operating on the same procept, which can be decomposed into process for calculation purposes whenever desired.
>
> (p.135)

*

The group offered mixed reactions to the notion of the 'procept'. Many had already confronted the idea and even used it in their own research. Some found it a dated idea, others found it useful in developing their own philosophies of mathematical thinking.

The main concern was the very word *procept* and how different it was from the very idea of the concept, and the mathematical concept. What were the subtle differences which Gray & Tall (1994) were hypothesising and offered apparently supportive evidence for?

Another criticism made by the group was that there was a lack of qualitative data. This might shed more light more on the written work of the children in the reported studies. Clear cut definitions of less- average- and more-able students were not adhered to in this context and many members of the group suggested a need to observe what the children were doing (saying) for themselves. Counting was viewed in a variety of other ways to those highlighted in the paper.

Reaction to these views arose in the group. Some believed that the authors were merely explaining a phenomenon in mathematical thinking which could help us understand more about the ways children are counting and doing elementary mathematics. In this light, reasons for successful thinking can be highlighted. The main issue of the paper, which was said by one member to have not been discussed with more subtlety, was the role of the symbol.

It is the role of the symbol which gives rise to procedural thinking and conceptual thinking, both separately and dualistically.
In summary, they group raised the questions below in reaction to the paper:

- What is a concept in mathematics, and what is it to think conceptually?
- How is this different from proceptual thinking?
- What is learning elementary mathematics and what can this paper offer us to help alleviate the problems in teaching it?
- What is the role of the symbol in mathematics?

*

Convenor's Endnote: The role of the procept in helping us to understand more about mathematical thinking is apparent in contemporary educational thinking. The subtlety of a mathematical action or process becoming conceived as mental object has and is still continuing to intrigue educational researchers. Gray & Tall (1994) have added to this view by providing evidence which supports their claim that the successful thinker does not just conceive these processes (action) as concepts(objects) but subtlety moves between the two in a dualistic cognitive process in solution. Whilst this is an elementary mathematical thinking (EMT) idea the notion does provide useful in understanding more about advanced mathematical thinking (AMT). There, the procept is abstracted further by the very nature of the progressive and abstractive nature of the symbol and the role of the symbol. One might find it difficult to apply to rings or fields, the symbol as a metaphor does appear to be a more advanced idea (Tall, 1998). But what is an important idea is, whatever the aesthetic nature of the symbol, what role does it play in AMT. This idea will be developed at the next meeting of the group.

Meeting held at University of Leeds. November 1998

One vital issue, which arose from our contentious discussions at the last meeting, was the role of symbol in undergraduate mathematics. This meeting debated issues concerning the role and nature of symbol in advanced mathematical thinking.

The meeting provided a forum for two purposes:

i) discussion of the main purpose and role of symbol from three theoretical perspectives; two from outside educational research,

ii) short presentation by Phillip Kent, Imperial College, concerning issues of symbolism and representations.

The first perspective we looked at was from a psychological/ anthropological perspective. We looked at Ed Hutchins' work, *Cognition in the wild,* which addresses the issue of symbols with a pschyo-social dimension. He accounts for internal symbol processing as a process of manipulating the physical environment into internal representations.

> With experience we lean about the regularities of the world of external symbolic tokens and we form mental model of behaviours of these symbolic tokens that permit us to perform the manipulations and to anticipate the possible manipulations. With even more experience, we can imagine the symbolic world and apply our knowledge, gained from interactions with real physical symbol tokens, to the manipulations of the imagined symbolic worlds.
>
> (Hutchins, 1996, p.292-

3) Hutchins continues to develop a more cultural form of cognition and in criticising pure cognitive science he describes how most problems in this field are consequences of our 'ignorance of the nature of cognition in the wild'.

> I believe that humans actually process internal representations of symbol. But I don't believe that symbol manipulation is the architecture of cognition.
>
> (p.

370) In supporting his claims he develops a way of studying cognition in the wild which he entitles 'cognitive ethnography'.

We then looked at the work of Terence Deacon, renowned researcher in neuroscience and evolutionary anthropology. Deacon's book, *The Symbolic Species* is a rich text on symbol and its evolutionary development. Whilst we could not offer the book sufficient time and analysis, a few issues are quoted here to offer a flavour of Deacon's philosophy.

Deacon explains how whilst it is important to accommodate a conventional set of markings ("signifiers") and objects, symbols, states of things ("signified") a more important issue is understand the way in which we refer between these things.

> A more complicated terminology is necessary then, to begin to differentiate between the way that words, as opposed to laughter and non-language signs, refer to things.
>
> (Deacon, 1997,

p.62) For a deeper understanding we must develop a symbolic competence. Deacon highlights Pierce's three categories of referential associations: icon, index, and symbol in order to assist analysis. An *icon* resembles something, e.g. landscapes, an *index* is causally linked to something else, e.g. a thermometer as an *indicator* of temperature. Whilst, *by a symbol*, one acknowledges some social convention which establishes a link between one thing and another, e.g. a wedding ring symbolises a marital agreement.

Deacon highlights a sensitive approach for differentiating between signs and symbols and interpreting, which might well be accommodated in the field of mathematics education.

> This demonstrated one of Peirce's most fundamental and original insights about the process of interpretation: the difference between different modes for reference can be understood in terms of levels of interpretation. Attending to this hierarchical aspect of reference is essential for understanding the difference between the way words and animal calls are related.

In mathematics, we have signs and symbols and the way we interpret them and differentiate them is important. These alternative works might well help us investigate the symbolic manipulation in advanced mathematics.

We then turned to the field of mathematics education and explored the work of Anna Sfard and Liona Linchevski who wrote on algebraic symbols and the gains and pitfalls of reification. Their work was carried out within a framework of the theory of reification, which addresses inherent process-object dualisms in most mathematical concepts. The theory explains how for most mathematical concepts an operational (process-orientated) conception emerges first, and mathematical objects which can be operated upon flexibly (cf Gray & Tall) are developed through reification of the processes.

In discussing algebraic symbols Sfard & Linchevski (1994) discuss the nature and growth of algebra throughout mathematical history. The development is explained in terms of transitions from operational to structural thinking but highlight how the transitions are important. What we learn about how we manipulate symbols in algebra can be explained in terms of how through history the role of symbol has changed. In individual leaning the authors offer evidence about the transition from purely operational algebra to structural algebra 'of a fixed value' and from there to functional algebra of a variable.

In criticism of how algebra is taught today reflection on the epistemological evolution of algebra is made:

> The curriculum literally reverses the order in which algebraic notions seem to be related to each other, the order in which they developed through the ages. The advanced structural approach is assumed at the outset even though the student is evidently not ready yet to grasp the idea of processobject duality.
>
> p.120

The final work we discussed was more in line with the epistemology of advanced mathematical thinking today as a paradigm. The benefits of such ideas were discussed in earlier meetings.

The group did grapple with the work of Deacon and its semiotics influences but it was apparent how the nature and role of symbol in mathematics is not clearly understood and how such ideas as signs, as differentiated from symbols, play alternative roles. This might well be especially pertinent when one looks at symbol in the context of "the wild" or symbols as physical representations.

Debate was made about what is meant by the *nature* of icons, symbols, etc. and whether investigation into understanding about the nature of symbol might benefit us.

Discussion about the role of symbol lead into discussion about the meaning of metaphors and how these relate to symbols. Symbols as metaphor are especially used in advanced mathematics and the issue might well be discussed at later meetings.

Sfard's work highlighted the evidence of a transition hierarchy in learning. Such findings were likened to Piaget's work but there was a notable concern about how such work was overtly observable and utilisable in the mathematics classroom. It was questioned: where in the reality of the child's learning world was a reification of meaning of symbols evident?

The group then heard a short discussion by Philip Kent, on symbol systems and AMT. A brief personal account of his talk follows:

Some **thoughts on Symbol** Systems **and Advanced Mathematical Thinking**

I presented some ideas drawn from writings by Andy diSessa (forthcoming) and Uri Leron (1987), which offer somewhat unconventional views on abstraction, mathematical thinking and learning, and the role of symbol systems. I selected from diSessa a discussion about Galileo's "laws of uniform motion" (dating from the early 17^{th} century), demonstrating the extraordinary power of symbolic algebra (which dates in its modern formulation from the *late* 17^{th} C.) compared with Galileo's geometrical techniques:

> We've redone a significant piece of work by one of the great geniuses of Western science, with amazing ease .... What we did would constitute only an exercise for a ninth-grade mathematics student. That, in fact, is the key. Galileo never had ninthgrade mathematics; he didn't know algebra! There is not a single "=" in all of Galileo's writing.

The effectiveness and accessibility of a symbol system rests on the degree to which it "expresses the right things", and one element of that is "picking the right level of abstraction", which captures the *relevant* details of a problem. The issue of relevance highlights the fact that the "right" level of abstraction is a function of both *who* is trying to solve a problem, and for what *purpose.* In these terms, Leron unfavourably compares the mathematical community's treatment of abstraction with that of computer scientists: "Though everybody agrees that mathematics deals with abstractions, the topic is not normally considered as part of mathematical subject-matter proper ... the opposite is true in computer science, where abstraction is a methodology which is explicitly discussed, taught and practiced". Further, Leron challenges the idea that students must necessarily tackle a mathematical topic armed with only the "official" level of abstraction, that, instead, it may be appropriate to devise a sequence of "intermediate level abstractions", none of which correspond to the whole mathematical truth, but which build towards it by highlighting relevant details whilst suppressing others for consideration later in the sequence.

*

Philip's talk led us to discuss a new theme, which will be explored next session. Symbols play different roles when one is to compare a novice mathematician's work to that of an expert's. It was expressed that Professor Leone Burton of Birmingham University might be invited to officially speak to the group on this issue at the next meeting. And so the debate continues.

References

Deacon, T. (1997) *The symbolic species: the co-evolution of language and the human brain.* Allen Lane: Penguin Press.

diSessa, A. (forthcoming), Chapter 1 of *Minds* + 1: *Computational media and new intelligence* [download from http://www.soe.berkeley.edu/ -boxer/papers.html]

Dubinsky, E. (1991) Reflective Abstraction in advanced mathematical thinking. In D. Tall (Ed.) *Advanced Mathematical Thinking,* 95-123. Dordrecht: Kluwer.

Gray, E. (1991) An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational Studies in Mathematics,* 22, 551-574.

Gray, E. & Tall, D. O. (1994) Duality, Ambiguity, and flexibility. In *Journal for Research in Mathematics Education,* 25 (2), 116-140.

Hutchins, E. (1996) *Cognition in the Wild.* MIT Press: Cambridge

Leron, U. (1987), Abstraction barriers in mathematics and computer science. In J. Hillel (ed.), *Proceedings of the Third International Logo and Mathematics Education Conference.* Pages 12-26.

Sfard, A. (1989)Transition from operational to structural conception: The notion of function revisited. In G. Vergnaud, J. Rogalski & M. Artigue (Eds.), *Proceedings of the 13th International Conference for the Psychology of Mathematics Education,* 151-158. Paris, France.

Sfard (1991) On the dual nature of mathematical operations: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics,* 22, 1-36.

Sfard, A. & Linchevski, L. (1994) The gains and the pitfalls of reification - the case of algebra. *Educational Studies in Mathematics,* 26: 191-228

Tall, D. O. (1998) Plenary given at the *International Conference for Teaching Mathematics.* Samos, Greece.

Thurston, W. P. (1990) Mathematical Education. *Notices of the American Mathematical Society,* 37, 844-850.