RELATIONS BETWEEN TEACHER'S REPRESENTATIONS AND PUPIL'S IMAGES

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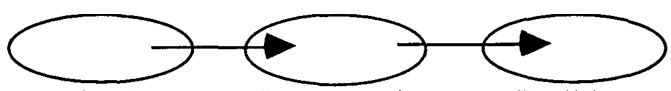
Abstract: This paper presents a simple constructivist model of teaching and learning characterised as a mapping between the set of representations that the teacher uses and the set of images that learners form. Data, collected in a phenomenographic study of primary school pupils' images, formed in consequence of their interaction with representations of two digit numbers, is discussed. Some implications for teaching are considered.

Introduction

My initial research interest was in the teaching and learning of fractions. Preliminary observations in Y2 and Y3 classes of a Solihull primary school, however, led me to the opinion that the difficulties that learners face in developing their conception of fraction is a particular instance of the general problems involved at each stage of expanding their concept of number. At any time when there is a need to introduce a larger set of numbers than pupils have previously experienced teachers engineer the encounter and pupils make sense of that encounter.

The 'engineering' involves the teacher using a variety of representations to communicate to pupils the mathematics that exists in the teacher's mind. The 'sense' the pupils make of their interaction with these tangible embodiments involves forming a mental representation which they can use in thinking about the mathematics. The mental representation takes a variety of forms and is often confusingly called image (it need not be visual). Here the word image is used in the mathematical sense i.e. a transformation from the representation that has been given. This provides a model of teaching and learning:



It allows for the possibilities of one-to-one, one-to-many, and many-to-one mappings between these sets. In this paper I describe my research on the representations and images. I will start with an indication of my method then consider each component of the model separately. In conclusion I consider the use of the model to describe my research process.

Method

The data gathering strategies that I have developed during the preliminary observations period have been appropriate for my purpose of describing and analysing pupils' responses to their experiences of the representations. Field notes of the lessons record:

- Teacher's use of words, actions, writing, drawings, equipment and the tasks they set;
- Pupils' response to teachers' questions, their questions to the teacher, their behaviour in response to tasks set;
- Pupils' answers to my questions during the lesson on how they are thinking about the tasks set.

Semi-structured interviews with individual pupils are audio tape recorded and notes made on their physical activity which relate to their descriptions of images. Initial questions are typically on relations between, or operations on, numbers with follow-up exploratory questions of the form "How did you decide that?", "What was in your mind when you did that?", "How do you think about that sort of thing?".

My model of teaching and learning is in the constructivist tradition, my assumption that pupils make sense of the mathematics by forming and using their own images is in the constructivist mould. My approach to the research is a naturalistic qualitative one which can be termed "phenomenographic" (Marton, 1988). Phenomenography is a research approach originally developed by researchers in the Department of Education of the University of Gothenburg. A phenomenographic study is an investigation of people's understanding of phenomena which seeks to categorise and explain the qualitatively different ways in which people think about the phenomena. The categorisation of descriptions is based on structurally distinctive characteristics of the responses in semi-structured interviews. The categories developed in one context are potentially part of a larger structure of categories applicable in other contexts. The initial discovery of previously unspecified categories of thinking may be peculiar to the researcher and context but the test of their validity is in their applicability for other researchers and as a source of explanation of differences in learning outcomes.

I start with the assumption that the images formed by pupils can not be studied in isolation from the context of the classroom or the interaction between the pupils and the representations used by their teachers. It is essential that I observe, first hand, the common experiences of the learners as a basis for my analysis of their different conceptualisations. My focus is on what the teachers say and do and how the pupils respond to this by developing images based on the representations.

Representations

The term 'didactic transposition' (Kang & Kilpatrick, 1992) is used to indicate that the representations that teachers use are not *the mathematics* but a transformation of the mathematics into a communicable form. The types of representations that are available to teachers have been identified as: real world contexts; manipulatable models; pictorial; spoken language; written symbols (Lesh, Post, & Behr, 1987).

In the lessons I have observed the representations used by the teachers tend to be influenced by the textbook writers. The teachers prepare the pupils to attempt exercises which are specific to a representation used in the books. I have focused on the teachers' attempts to extend the number sense of the children beyond their first experience of number as the easily countable cardinal number of a set. A variety of representations have been used:
Objects to count into bundles of ten (cubes, straws, marbles in bags),
Dienes arithmetic blocks,
Verbal (counting on and back in powers of ten, chanting emphasising place value),
Column notation (individual digit cards, numerals written in columns),
Number line, number track, number square,
Money,
Spike Abacus.

Two associated procedures are in evidence: NlO (counting on by tens - ordinal) and 1010 (decomposition in tens and units - cardinal) (Beishuizen, 1993)

Field notes give a flavour of the explicit representation-specific language of the teacher, T.
17/10/97 Y2- Developing 1010 strategy using Dienes blocks:

> T gives Mandy 2 tens ('Another way of putting it ?' M says 'twenty') and 4 ones ('How many altogether ?' M says 'twenty-four'). Gives Nina 1 ten and 2 ones ('How many altogether ?' etc.). 'Now put them together in my hands' M and N put the *tens in one* hand and the *ones in the other*, 'How many altogether? Look how easy it is to add them instead of all individual cubes.'

27/2/98 Y2 - Some pupils demonstrate in a revision lesson that the separating of tens and ones has become part of their way of thinking:

> 'Now add 20p on. How will you do that?' Ann says 'see what the first number is and add 2 on to it' another pupil says 'add ten on then another ten', another says 'if you had 59 take the 9 off and add the tens on then add the 9 back'.
>
> The teacher reinforces 'what I don't want you to do is to do 39,40,41, like that, 'cos that would be silly.'
>
> Subsequently a variety of methods are in use for adding 20p on to different amounts of money including John counting on in ones using the number square and getting them wrong through miscounting.

Mapping from representation to image

The pupils listen, watch, and attempt the tasks set. From this activity they form their images. I follow Lesh and Kelly (Lesh & Kelly, 1997) in thinking that:

> Humans interpret their experiences using internal conceptual structures, which cannot merely be received from others, but which must be developed, actively, by each individual. Further we assume that the meanings of these constructions tend to be partly embedded in a variety of external systems of representation, (p398)

They, like Mason (Mason, 1987) , reject the conveyance metaphor of teaching and learning i.e. that meaning is carried by representations and the learner must simply take out the meaning when the representation is received.

If there is a hierarchy in the representations it is difficult to detect in the textbooks or from classroom observations. Thus it seems that these mathematically naive learners meet a heterarchical collection of representations and can 'actively develop' discrete images. Each representation has procedures which the teacher demonstrates to pupils whose subsequent successful performance of these representation-specific procedures can be misinterpreted as understanding of the underlying mathematics.

It is possible for the pupils to adopt each representation-specific procedure separately without recognising the common mathematics i.e. without developing their concept. They add tens by: saying next word in word sequence; going down a column on a number square; taking a ten-step on a number line; replacing tens digit with one higher; adding one to the tens column; having an extra ten-block; having an extra lOp coin. Many find one representation easier to use than others so use it exclusively and cannot explain one representation's procedures in terms of another. The representations and their procedures are sufficiently different in appearance to seem like different concepts.

The teachers observed have provided representations in the attempt to enhance previous images and have helped pupils re-collect the experiences that have given rise to them, in order to develop more efficient procedures. However the shared image that was the teacher's goal was not necessary for effective completing of exercises. Many pupils were reluctant to abandon previous dominant images (e.g. addition by counting-on in ones) which had served well in the past. It could be that there is less expenditure of mental energy needed to use a slow previous procedure than to drive the imagination required to form a more robust image. The desire for efficiency, whether due to enculturation or instinct, may determine that pupils stay with the tried and tested rather than invest the effort required to change. The frequent recall of previous images to aid the formation of new ones could also have the effect of strengthening them.

It is apparent that there is not a one-to-one mapping from the teacher's image to pupils' images mediated by representations. The teacher may choose a variety of representations that embody the same mathematics but each medium can become a separate message.

Images

(Russell, 1956) gives a definition:

> An image is a centrally aroused experience which reproduces in part some previous perceptual experience in the absence of the original sensations. (p68)

where 'perceptual' is taken as a reaction to any stimulus i.e. not only visual. The images, though, are *likely* to reflect the experiences that give rise to them (Lawler, 1996). It is also suggested that a learner's "concept image", consisting of mental pictures, properties and processes, may develop with each new experience (Tall & Vinner, 1982).

I have used the word 'image' to mean that which the learner makes use of in his thinking as a result of his interaction with the representation. In interviews pupils may not mention a teacher's representation but it could still have left a 'trace' (akin to construction lines, or the trace left after an object has passed, or the trace element that is so watered down it is almost undetectable)
which influences the development of their image. The image developed by learners as a result of exposure to representations are more-or-less accurate re-collections of their experiences transformed by their imaginations. I take 'imaginations' in this sense to mean the manipulation of images into a form which the learner finds comprehensible. The image formed may lose many of the subtleties of the original representation as it is moulded to fit with the learners dominant preVIOUS Images.

In pilot semi-structured interviews I have encouraged pupils to describe what they are thinking when they add 53 to 24, how they decide what comes after 276, how they work out 86p-12p. I have explored a variety of categorisations :

- Recency - Some pupils make explicit reference to a recently introduced representation and associated procedures, others use strategies they have developed themselves, others call up their most dominant previous image;
- Modality - Some pupils describe "seeing" blocks or squares or numerals, others "hear" words in their own or their teacher's voice;
- Variety - Some pupils use the same representation for each question others use a different representations for each;
- Progression - The majority of pupils interviewed early in Y2 used counting on and back in ones whilst Y3 pupils predominantly separated tens and ones and made reference to written algorithm procedures;
- Completeness - Some pupils made reference to a representation but could not recall the procedures;
- Flexibility - Few pupils could use more than one image for a particular question.

Exerts from transcripts of interviews with Y2 pupils conducted in October when they had been using both number squares and Dienes blocks give an indication of the range of images:

1) Hazel has developed a strategy that is neither 1010 nor NlO and she makes no reference to any of the representations that have been used in class recently. My assumption that she would use one of these means that my questioning and prompting are representation-specific. The first question, which was designed to see if she would use a taught procedure, was interpreted as a request for her to calculate in a way she chose, perhaps because she had no image to support the suggested procedures. The fact that I failed to persuade her to use an image based on one of these representations could indicate that she has not formed one because her own is sufficient for her needs. She has not incorporated the teacher's representation-specific procedures into her image because they do not fit with it.

I Do you remember how to do those? (points at sum, 24 + 53, printed on paper)

H gave answer 77 after moving fingers for 27 seconds

I When you added those you were counting with your fingers weren't you. Can you tell me what you were counting?

H I counted 53 and 24.

I Good so you started with the big number and then you added 24. Did you add 20 first or 4 first or did you count on 24?

H I added 5, 5, 5, 5, and then the 4.

I Would you have liked to use the 100 square or the rods and blocks to do that or did you like doing it like that in your head?

H I like doing like that.

I Can you write the answer down for me?

H Forgotten the answer.

H starts counting again saying the numbers semi-audibly, using just one hand, counting on five at a time remembering how many fives she had added and got the same answer in 20 sees.

2 Neal has an image of 2-digit addition that uses the procedure requiring the digits to be separated but does not have the procedure which adds tens to tens and units to units

I If you had to do this in your head what would you do?

N You get 24 in your head then add on the 5 and the 3.

I So what would that give you then?

N 32. (8 sees)

3 Nina has some recall of the procedures related to the Dienes blocks but the overload caused by attempting to manipulate this image means that she can not make use of it:

I Last week you were doing these (points to sum). You were doing them with rods and blocks and you had been doing them with a hundred square but if you had to do that in your head could you do it? Would you like to try?

N ... Don't know. (the answer)

I What would you start with?

N 24 .. and then you would add 53.

I Would you like to try it

N ... Don't know. (the answer)

I If I had brought the rods and blocks and the hundred square which would you choose?

N Blocks

I You like using those best. If you had the blocks what would you do with them.

N I would put 24 and 5 big ones.

I And then you would be able to put them together. Can you imagine that? If there were 24 and 5 big ones can you work that out?

N ... don't know.

Implications for teaching and learning

If learners are to develop their concept it is insufficient simply to have the ability to recall representation-specific procedures in response to representation-specific questions. I suggest that four steps are required: Re-mind, Re-collect, Re-cognise and Real-ise.

In order to make a new representation meaningful (related to other representations) the teacher has to re-mind pupils of the other representations - she encourages them to bring back to their minds those previous encounters with this mathematics. Pupils need to re-collect those images that they have previously formed then to re-cognise that their new image is related to the others. The act of re-cognition implies a sense of new understanding of this and previous images in the light of the links between them. Finally real-ise signifies the act of making real. A learner only realises what mathematics is conveyed by a representation when they have an image of it that is real for them.

Conclusion

As a naive/novice learner in this field I am attempting to build images from perceptions and representations of the learning process. Since I can only understand when I can attach an image to other images that I already have I may resort to analogies to give me an opportunity to recognise. In order to communicate part of my understanding of teaching and learning I give this representation which I refer to as "Representations and Images" using words that symbolise my image. In this act of didactic transposition I inevitably transform the image and again the medium becomes the message.

The purpose of the study is to present a representation of teaching and learning that might provide an image for other teachers to use.

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