CONVENTION OR REALITY?

M. M. RODD & M. R BARBER

The OPEN UNIVERSITY & THOMAS TELFORD SCHOOL *

*from Autumn 1998: Leeds University & Cockshut Hill School, Birmingham, respectively.

Abstract*: this report both introduces to a mathematics education audience some research in contemporary philosophy of mathematics and develops the relevance of such research to mathematics in education. The philosophical theories to be considered are those of the physicalistrealists, specifically in this paper, Resnik and Bigelow; physicalist-realists offer theories of how mathematics is, in some sense, part of the physical world The notion of an 'instantiation' of a mathematical theorem, or object, is distinguished from a more general and familiar 'representation' in order to help conceptualise the possibility of non-linguistic experience of mathematical relationships or entities. The question of relevance to mathematics in education is approached by presenting mathematical activities to engage with and school-texts to analyse.*

Introduction

Our objective, in this session, was to 'report' on work on realism in the philosophy of maths through workshop-style activities and discussion. The wider aim was to come up with an appropriate sense of the distinction between 'real' and 'conventional' by working with an example from the secondary school curriculum. The motivation for the research came from our sense, as mathematics teachers, that there is a difference between mathematical conventions (data representations, particular measures, rules for rounding up, for example), which are properly part of the school mathematics curriculum, and aspects of mathematics, also within the curriculum, the truth of which is independent of mathematical conventions people have established through the ages (for example, the existence of non-ratio-like numbers which we consider, below). But what is a suitable conceptual distinction between convention and reality as they pertain to the school mathematics curriculum?

To answer this question, we need to conceptualise 'convention' and 'reality' in broader terms, then see where parts of the curriculum sit (we have already exemplified this distinction just in order to clarify the basic question). The 'conventional' category is essentially the linguistic-cultural aspects of mathematics. And is currently the dominant conceptualisation of the nature of mathematics in education. In order to follow though on our practitioner's sense that there was something other than humanly constructed conventions, we looked for theoretical underpinnings for the notion of independent mathematical reality. And then we interpret this theory for the mathematics of the school curriculum.

Philosophers who hold that there are convention-independent mathematical truths are called 'realists' in the philosophical register. And their conception of the nature of mathematics offers different insights from their contraries, termed 'anti-realists' by Michael Dummett. Anti-realists also include 'conceptualist' philosophers, like Rorty (1990) and Bloor (1976). The views, at least of the latter, have been interpreted in the mathematics education domain by Ernest (1991, pp 45-58).

The theoretical underpinning we put forward was based on 'physicalist-realism'!. Physicalist-realists claim that mathematical entities do exist and their reality is bound to human beings' particular physical make up within the physical world. Although a sense of the 'objectivity' of mathematical entities *can* be explained without advocating this sort of reality (see, for example, Ernest, 1997), we choose to grasp the nettle of the 'objectivity' of mathematics directly by this realist ontology. The specific philosophers we quote here are Michael Resnik (1993, 1998) and John Bigelow (1988).

A brief introduction to *realism in mathematics*

Realism in mathematics, within philosophy, concerns ideas about how mathematical 'abstractions' may indeed exist. There are many forms of realism, from Plato's conception of transcendent 'Forms' to Putnam's 'internal realism'z (e.g. Putnam, 1990). However, a collective feature of realist theories of the world is that "existence is prior to theory" (Harre, 1986, p5). In terms of contemporaries, Harre's 'modest realism' recognises that securing a scientific belief is, in some sense, a social activity. Nevertheless, Harre insists that "for there to be public reliability something must exist independently of whomsoever first found it." (pI2). Putnam eschews conceptual relativism like the Rortarian 'conceptualist' position[3]. Popper, too, was also a realist, (e.g.,1972). His metaphor of the 'mountain beneath the clouds' suggests a 'reality' which 'we seek'. In a nutshell: to assert 'realism' involves asserting the existence of an external world and it is that external world which, in theory, is the ultimate arbiter of truth values. Physicalist-realists in mathematics furthermore assert that mathematical existence presupposes a material reality which is the universe of which we are a part. In mathematics educationalliterature[4], however, such positions have been over-shadowed in recent times by versions of 'conceptualism', which, roughly, sees mathematics as a language or social practice, (see Sierpinska and Lerman, 1996, for a review).

1 Irvine (1990) defines this type of realism thoroughly.
2 This position, briefly, denies the 'God's eye view' of nature: Putnam claims to be a 'small *r'* realist (Putnam, 1990). 3
"Rorty' s view is just solipsism with a 'we' instead of an I" (Putnam, 1990, pix)
4 At the June 1997 BSRLM, one of us, MMR, gave an introduction to Penelope Maddy's realist views, (see Rodd, 1998)

Resnik and Bigelow

Michael Resnik's view is that mathematics is a general science of patterns and structures (i.e. discovered results about patterns and structures). So, for example, whether a number is prime or not is a physical property of arrays of units (whether they can be made rectangular):

> "mathematical objects are abstract entities existing independently of us and our constructions and theories ... (most of) the claims of contemporary mathematics are true, and they are true independently of our holding them to be true ... mathematical reality transcends our own existence, beliefs and experience."(Resnik, 1998, p319)

> "We can never study a mathematical object, such as pi, in isolation, as we might study a specimen. Instead we study the patterns or structures in which mathematical objects are positions"(ibid., p326)

Resnik denies that any part of mathematics is directly perceptible, (I 993, p50), but that mathematical entities are posited by limiting actions. His paradigm example being that of a geometric point, which is a posit derived from repeated cutting on a line.

John Bigelow's view is that mathematics is the theory of universals, i.e. a systematic collection of statements about relations among universals[5]

> "Natural numbers are, I urge, universals which are instantiated directly by individuals in the world; but the other kinds of numbers cannot be construed as properties or relations of individuals, but rather, must be construed as *relations between relations,* using a technique pioneered by Frege and Whitehead and Russell" (Bigelow, 1988, p5, emphasis in original)

> "sets as universals of a special sort, ... [for] all the structures mathematicians have hitherto wanted to study are *instantiated* by sets" (ibid., p6)

> "And since we are physical beings ourselves, through and through, the same mathematical patterns that are in the world around us are also present inside us" (ibid., p2)

Bigelow's paradigm mathematics is that of the irrationality of .../2. We exploit his specific instantiation of .../2 to show the a-conventional existence of non-ratio-like numbers, below. Thus these two philosophers offer ways of locating some mathematics within the reality of the physical world.

Instantiation

Resnik, as well as Bigelow, uses the term 'instantiation':

5 "universals are the properties relations, patterns, slructures and so forth, which can be shared in common with many diverse individual particular things" (ibid., p 11), for example, 'redness', '17ness', 'roundness', 'non-ratio-ness'

> "templates as concrete representations of patterns ... templates need not instantiate the patterns they represent; written musical score, for example, are templates for sound patterns"(Resnik, 1998, p326)

Understanding the term 'instantiation' should help in the conceptualisation of a distinction between convention and reality in mathematics: an *'instantiation'* is a special sort of representation which 'is an *instance* of the concept in question. Bigelow, above, was quoted as saying that numbers are the universals which are instantiated by individuals. A way of grasping the idea of physicalist-realism is to look for instantiations of mathematical theorems and to distinguish these from other sorts of representations (which are often cultural constructions). Instantiations are a sort of representation but a distinguished class which DIRECTLY represents the entity, relation or theorem.

We offered an example of an instantiation for participants:
Activity: to find an instantiation ofa square root algorithm (Devi, 1977, p71 ,Wheeler, 1974 p79) We start with a HTU square number:

What is the square root of 289?

289 = 2xO + 8xl + 9 (where '0' stands for a one hundred square, and 'l' stands for a ten strip)

The idea is to arrange the Os and the l s and the units in a square. Then the root is just the length of the side. How can this be expressed algorithmically? Then extended for larger numbers and for approximations to roots of non-square numbers?

Given this conception of instantiation, the next activity was to 'instantiatiate' the non-ratio nature of -- ./2. If this can be achieved, we claim that this shows the 'physicalisticaUy real' existence of an irrational number - a discovery, in short, rather than (just) a linguistic construction. The method for such an instantiation is worked through in Bigelow 1988, p33. The idea is to show that no square grid of pebbles - discrete, easily manipulable objects of perception - can ever be rearranged to make two EQUAL smaller squares. Thus the non-existence of two natural numbers such that $n^2 = 2m^2$ forces the real existence of a non-ratio-like (a.k.a. 'irrational') number.

Application to *school mathematics*

Participants were given a selection of extracts from school maths texts (Bostock et al. 1993, Hackney et al. 1994) to scrutinise with the objective of observing and c1assitying where the texts presented irrational numbers as a 'real phenomenon' and where they presented them as an invented construction.

As an example of these extracts, here is a part of Hackney *et al.* :

> "We know that ..J2 is about 1.414. But the Greeks had no calculators, and no decimal numbers. The only way they could talk about numbers like this was in terms of fractions that is, as ratios of whole numbers. They could say that one and a half was i. But they could not find any ratio of whole numbers that was exactly equal to ..J2, which was not surprising, because there is no such ratio .. ,. Some people say that Pythagoras came to a sticky end because he invented irrational numbers." (p91)

Tn usual practice currently, the topic of irrational numbers is one in which the learner completes the GCSE-Ievel study without knowing what an irrational number *is.* So, as an application of being able to distinguish between 'convention' and 'reality', we suggest that it may be helpful to the learner to have the teacher clarifY the conventions-reality status of a new topic, even if there'll be a lot more to the topic in future study.

Conclusion

One of the fascinating thing about mathematics - from our perspective - is the way conventions do combine and extend mathematical reality: abstraction from a perceptual or experienced phenomenon (e. g. perception of discreteness or relations of kinship, respectively) has both scientific and systemic features.) Furthermore, on this science/language distinction: one is not always able to decide easily, for a given mathematical topic, whether it is an encoded aspect of the world (which was discovered) or a projected convention which was (implicitly) agreed within some group of language users. Abstracting (formulating key features), axiomatising (defining conventions), discovering (finding out: either about things or consequences of the defined conventions) and modelling (applications of any of these) are, in our view, *all* important aspects of the mathematical enterprise. This enterprise is rooted in the physical experienced world and grows from there in receptive and nourishing cultures.

To sum up: the relevance of realist philosophy of mathematics to school mathematics has been exemplified and developed through the case of irrational numbers:

1. we claim that irrational numbers exist as real entities which can be instantiated physically though not directly observed (as no measured length 'measured up' as an irrational)
2. the encounter students have with irrational numbers <u>either</u> tends to represent them as lengths which can never be measured completely accurately <u>or</u> as algebraic symbols, ..J2, for example, which are manipulated algebraically.
3. so the question, which is a philosophical one, of the essence of an irrational number, is relevant to education. This is because, in the school texts we have found in our research, the concept of an

irrational number is dealt with 'conventionally' (i.e. use so many decimal places - 1.414 - or use this symbol ·.J2 for the length of the diagonal of the unit square) rather than as a 'real' entity on which the conventions (of truncation or symbolic representation) have been laid to help with communication of and work with of these numbers in applications.

references

BIGELOW, 1. (1988) *The Reali~y o.lNumbers* (Clarendon Press, Oxford)

BLOOR, D. (1976) *Knowledge and Social Imagery* (Routledge and Kegan Paul, London)

BOSTOCK, L. *et al.* (1993) *STP Mathematics 5A* (Stanley Thorns, Cheltenllam)

DEVl, S. (1977) *Figuring* (penguin, London)

ERNEST, P. (1991) *The Philosophy o/Mathematics Education,* (Falmer Press, London)

ERNEST, P.(1997) 'Texts and the Objects of Mathelllatics' *Philosophy of Mathematics Education Journal* no. 10 (electronic version)

HACKNEY, 1. *et at.* (1994) *Heinemann Mathematics Upper Course* (Heinemann Educational, Oxford)

HARRE, R (1986) *Varieties of Realism* (Blackwell, Oxford)

IRVINE, A. D. (1990) (Ed) *Physicalism in Mathematics* (Kluwer, Dordrecht)

POPPER, K. (1972) *Objective Knowledge* (Clarendon Press, Oxford)

PUTNAM, H. (1990) *Realism with a Human Face* (Harvard University Press, Cambridge MA.)

RESNIK, M. D. (1993) 'A Naturalised Epistemology for a Platonist Mathematical Ontology' in RESTIVO, S. et al. (Eds.) *Afath worlds: philosophical and social studies of mathematics and mathematics education,* (SUNY Press, New York) (pp. 39-60)

RESNIK, M. D. (1998) 'Proof as a Source of Truth' in TYMOCZKO,T (Ed.) (new edition) *New Directions in the Philosophy of Mathematics* (Birkhauser, Boston)

RODD, M. M. (1998) 'Maddy's Mathematical Reality' *For the Learning of Mathematics,* 18,2 pp.34 - 41

RORTY, R (1990) *Objectivity, Relativism and Truth* (CUP, Cambridge)

SIERPINSKA, A. and LERMAN, S. (1996) 'Epistemologies of Mathelllatics and Mathematics Education' in *The Handbook for Research in Mathematics Education* Vol. 2 (Kluwer Academic Publishers, Dordrecht) (pp 827 878)

WHEELER, D. (1974) *R is/or Real* (Open University Press, Milton Keynes)