# TWO TYPES OF MENTAL ARITHMETIC AND THE EMPTY NUMBERLINE

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*The empty numberline (ENL) was introduced as a new model in Realistic Mathematics Education (RME), after discussions in the Netherlands how best to improve the basic skills up to 100. Studies like our first National Evaluation Test* (1987) *pointed to a possible imbalance between two types of mental arithmetic: much emphasis on mental strategies may diminish practice in mental recall of basic number facts. In 1990 the Freudenthal Institute (Utrecht University) set out the publication of a more balanced view, inc01]Jorating Leiden research into addition and subtraction up to 100. This background of the empty numberline might be relevant to British discussions today about (mental) maths teaching. A summary is given of the outline and the research outcomes of the experimental ENL-program in Dutch 2nd grades/British Year3. Apartfrom the positive cognitive results the ENL-model also stimulated pupils' own recordings of mental steps. Effects of a short experiment in a British Year3/4 class are briefly mentioned.*

## 1. Two types of mental arithmetic in a more 'balanced' RME-theory

The good performance of the Dutch 9-year-01ds in the recently published TIMSS report was an encouraging surprise in our country. After the introduction of an improved 'second generation of realistic textbooks' in the beginning 1990s, curriculum implementation moved high up on the agenda. Good teaching practice according to RME-princip1es is now one of the targets of teacher training and teacher in-service courses. For instance finding a better balance between whole-class introduction and interactive discussion of problems followed by individual and group work. Therefore, prof. Treffers of the Freudenthal Institute welcomed in a Dutch newspaper (Lange1aan, 1997) the good TIMSS results with the headline: "Long live who1e-c1ass maths teaching!"

During the 1980s realistic textbook design as a precondition for implementation was central in focus. The Wiskobas' group started in the 1970s with the publication of several projects, but schools asked for more longitudinal and coherent curriculum guidelines (Treffers, 1991a). The frrst realistic textbooks were published between 1981-1984, paying much attention to a variety of models for number and problem representation (Gravemeijer, 1994). One of the purposes was to overcome the one-sided use of structured apparatus in maths teaching, as the realistic movement emerged as an alternative to the influence of Structural Arithmetic and New Maths (cf. Stem, 1971). Instead of introducing formal structures like place value with concrete materials to children, Freudenthal (1973) advocated the more radical view of linking-up early maths activities to children's own informal (counting) strategies, and postponing the more formal aspects till later.

Interestingly, such arguments are also being voiced at regular intervals in the UK (Plunkett, 1979; Liebeck, 1984; Anghileri, 1995; Thompson, 1997). However, apart from a radical project like CAN (Shuard, Walsh, Goodwin & Worcester, 1991), they never did come into practice maybe because of the stronger influence of New Math + Piagetian psychology in British maths schemes? (Thompson, 1997). Anyway, the 'fIrst 1980s generation' of realistic textbooks in the Netherlands also could be characterized as somewhat one-sided because they emphasized very much a variety of models and mental strategies at the expense of daily practice in mental recall of number bonds etc. (having a traditional flavour at that time).

Evaluation studies like our fIrst National Evaluation Test of Primary Mathematics (1987) pointed to an imbalance between these two types of mental arithmetic. Therefore the Freudental Institute set out to publish - after a series of discussions at conferences and in articles - a more balanced 'Specimen of a national program for primary mathematics teaching' (Treffers & De Moor, 1990). The empty numberline (ENL) was one of the new proposed models, to replace the empty hundredsquare which had been introduced during the 1980s as an improvement over arithmetic blocks (Beishuizen, 1993), but turned out to be a rather diffIcult model for weaker pupils. The numberline as such, of course, is an old model. Its empty ENL-format, however, was a new feature stimulating several aspects of mental activity and mental arithmetic, as we will see.

## 2. Leiden research into mental addition and subtraction up to 100

Research at Leiden University concentrated on mental addition and subtraction up to 100 (Beishuizen, 1993). Compared to arithmetic under 20, procedural knowledge is going to playa larger role here, especially how to deal with the tens in two-digit numbers. Another argument was that many children begin to develop half-correct procedures in this domain, not only for written algorithms but also for mental computation (Fuson, 1992). For instance, one of the outcomes of the above mentioned National Evaluation Test (1987) was that only 55% of Dutch 3rd-graders (Year4) had suffIcient command of problems like 64-28 (subtraction with carrying).

Two different types of mental computation procedures are widespread in use. The placevalue oriented decomposition method, where tens and units are split off and handled separately (e.g., 46+23 via 40+20=60 and 6+3=9, answer 60+9=69). We call this split method for practical reasons of easy scoring with an acronym: '1010' (10+ 10 or 10-10). The second method is sequentially oriented on the number row and proceeds by counting in tens from the fIrst unsplit number (e.g., 46+23 via 46+20=66, 66+3=69). This jump method is called with an acronym:

'NlO' (Number+ 10 or Number-lO). Similar distinctions are made by Fuson (1992) for American and by Thompson (1997) for British children. Here, 10 10 is most frequently in use, as well as a variant which we call 'lOs' because the units are handled sequentially (40+20=60, then 60+6=66, 66+ 3=69). Probably the early emphasis on place value and on ten-based structured apparatus is an explanation for this popularity of 1010. For instance in the Homerton Library I found how Stern (1971), in her well-known book on Structural Arithmetic and New Math, is explicitly emphasizing 1010 because "the important principles of mathematics must be demonstrated" (p. 223). For a difficult problem like 88-49 "the New Math should not be afraid to introduce a simple step of algebra": (70+ 18)-(40+9)=(70-40)+(18-9), etc. (p. 225). Another explanation is that 1010 as a mental method is dealing with tens and units (separately) in a similar way as the written algorithms do, which still prevail in American and British textbooks.

On the other hand, in German and Dutch textbooks for a long time there has been a greater emphasis on NlO as the real mental method, taking less steps and less memoryload by carrying out the procedure in a sequential and more fluent way. The only thing is that children need regular practice in building up the lO-jump counting strings along the (inner) numberline (e.g., 16,26,36, 46, 56, 66, etc.). That is an obstacle for weaker pupils and many of them prefer 1010 as an 'easier' procedure. This 101O-preference, however, in the long run works out as a handicap, because for difficult subtraction problems including carrying (cf. above) 1010 is a more complicated procedure and more susceptible to errors than N10. It has been found in several Dutch studies, that more able pupils develop a preference for N10 while many weaker pupils hang-on to 1010. In one study we found that some pupils develop along a sort of 'procedural compression' (Gray, 1997) learning strand: 101O->lOs->NlO (Felix, 1992). In another study some pupils appeared to be capable of direct strategy change like 101O->NlO, when presented with nonstandard 'difference' or 'missing addend' problems (Beishuizen, Van Putten & Van Mulken, 1997). These transitions from lengthier into shorter procedures seem to have a similar (extra) significance, as the transition to counting-on in the development of counting strategies (Gray & Tall, 1994; Nunes & Bryant, 1996).

## 3. The new empty number1ine (ENU pro~m

The RME-argument in favour of initial learning of NlO is its link to informal counting strategies children bring with them to school. As Treffers (1991b) puts it: counting should not be suppressed but mastered. Therefore new models like the arithmetic rack (up to 20) and the empty numberline (up to 100) were designed to support and challenge such a gradual development or 'progressive mathematisation'. Counting level increases from counting-all to counting-on, and is abbreviated into steps of 2s, 5s, later lOs. This RME-view is in great contrast with the structuralistic view of Stern (1971) on "the disastrous habit of counting-on" because "arithmetic should not be based on counting" (p. xi). Treffers and De Moor (1990) have sketched the further development of NlO for a subtraction Problem like 65-38 at three levels of abbreviation: (1) 65-10-10-10-5-3, (2) 65-30-8 (3) 65-40+2. On this latter level we see a strategic adaptation (NlOC = Compensation) of the procedure. Another option is strategy change from Subtraction to Adding-on (38+2+20+5=65, answer 27, acronym: 'A 10'). 1010 is introduced later at the end of the 2nd grade (Year3) as the more formal place-value model and link to vertical algorithms introduced in the 3rd grade (Year4).

The empty numberline (ENL) -with a structured introduction- was designed as a more powerful model for inviting and stimulating the (spontaneous) development and leveling up of computation procedures and strategies up to 100 as described above. According to these RME-principles an experimental ENL-program was developed and implemented under two conditions in several 2nd-grade (Year3) classes (N=275). Context problems were mixed with number problems to foster flexibility of solution strategies. Drawing jumps on the empty numberline worked out very well as a natural way of keeping track and recording of mental solution steps. This proved to be a great advantage during the last 3 months of the ENL-program, where more time was spent on whole-class discussion. Many pupils no longer needed the support of the ENL, and they even used acronyms as 'labels' to describe their mental solutions. The teachers considered these metacognitive effects as equally or even more important as the cognitive imProvement Another feature is the open character of the ENL-modelleaving much room for individual differentiation in level and Preference of procedure use and strategy choice.

During the BSRLM paper presentation examples of pupils' work and test results were showed to illustrate these latter two aspects of mental arithmetic: 1) Proceduralization and 2) strategy development (see for examples: Beishuizen, in press). The research results are being published shortly (Klein, Beishuizen & Treffers, in press). One significant outcome was a strong improvement on subtraction problems. On a National Arithmetic Test (Cito-E4) for the 2nd grade (Year3), taken as an external criterion test at the end of the Program, problems like 64-28 now scored about 80% correct. This outcome also means that pupils' scores hardly showed a fall from addition towards subtraction, as mostly is the case in maths tests. A greater flexibility in strategy choice like adding-on as an alternative solution to some types of subtraction problems is one of the explanations (Klein, Beishuizen & Treffers, in press). As a sequential model the ENL makes the operations of addition, subtraction, adding-on, compensation, etc. as well as their possible interchangeability, much more visible and transparant than hundredsquare or arithmetic blocks do.

A short experiment consisting of 6 lessons with the ENL-model in a British Year3/4 class demonstrated this positive sequential modelling effect. On a pretest, nonstandard 'difference' problems were solved at a low level of 31 % correct (many 1010 errors), but improved to 75% correct on a (same) posttest due to ENL-support and quickly adopted NlO and AlO solution procedures (Beishuizen & Rousham, in preparation).

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