INVESTIGATING CIDLDREN'S INTUITIVE UNDERSTANDING OF NUMBER OPERATIONS BY FORMALISING THEIR MENTAL STRATEGIES.

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Abstract

Research has shown that young children's intuitive view of addition is non-commutative. This paper describes a 14 week study of a small group of 5 and 6 year old children inventing and using their own symbols to ask and answer questions about numbers in a 'tranformationally rich subculture. In a 'stepping stone' scenario, symbols referred to changes (transformations) of position rather than aggregations of objects. No reference was made to traditional mathematical terms such as addition, subtraction, equals etc but the aim was to develop a pedagogy whereby the children's intuitive system could eventually be 'grafted' onto the conventional system of arithmetic.

Introduction

The research was conducted with a small group of 5 to 6 year old children (Grade 1) in a small rural school in the Lake District (U.K.) over a period of 14 weeks. Based on the writer's personal experience in developing countries and also on research findings world-wide, the following broad presuppositions motivated the investigation:-

(i) Children begin school with a significant and valid core of intuitive notions about number, derived from their pre-school experience. They also possess a latent mental number ability which is considerably undervalued in schools. (Durkin, 1993; Klein & Starkey, *19'i57)*

ii) Children's understanding of *conventional* mathematical symbols is greatly overestimated - but children can invent and use their *own* symbols with great facility.(Gardner & Wolf, 1983; Hughes, 1986; Atkinson, 1992)

iii) 5 and 6 year old children's views about numbers do have coherence and validity within their own criteria - about which we know very little. As Sinclair & Sinclair *(19'i57)* argue, from a Piagetian standpoint,

' ... children construct concepts and collate information on their own, without receiving any specific formal instruction, and in many respects build up a numerical world of their own, which is often incompatible with what they are presented with once formal education starts.'

Background to the Research

Support for the idea that children use and are capable of creating their own mathematics have come from several cross-cultural studies, such as Gay & Cole (1967); Lave (1988); Saxe (1991); Carraher (1991). Also, Bishop (1991a, 1991b), has described mathematics as a 'pan-human phenomenon' and suggested that each cultural group is capable of generating its own mathematics. From this perspective the research can be regarded as a form of *acculturation* in which a relatively alien culture is introduced to the children.

Previous researchers have allowed children to develop (or 'invent') their own mathematics but then have suggested the conventional symbols to for children to use instead of their own. For example, Neuman (1987; 1993) suggested 'equals' and 'plus' signs to children at an early stage, to symbolize aggregative solution strategies they had arrived at in the context of measurement.

Children have also been encouraged, with some success, to invent and use their own mathematical symbolism in the 'emergent maths' movement (Atkinson, 1992). However,although this has provided valuable insights into children's understanding of symbols, the possibility of extending or formalising these symbols to form an 'intuitive' system does not appear to have been considered.

Other research dealing with children's use of signs has generally been made in the context of *aggregative* addition, where numbers referred to collections of objects or 'sets'. [Sets, as normally taught in a school context refer only to the cardinal aspect of number and are not considered by the writer to provide an adequate basis for all number situations.]

Theoretical perspective

What is the origin of children's intuitions; what is their relation to the operations as defined by mathematicians - particularly within the set-theoretic paradigm? Can we describe these intuitions as 'concepts', and if so, are they constructed by children themselves or are they culturally induced? These are difficult questions which cannot be answered at the present time. We can however begin to define the parameters of the problem by looking at children's learning through *symbols* (whether symbols for concepts, operations or intuitive models).

Constructivists have argued that mathematical concepts are (or should be!) constructed by children according to their needs, and question whether a symbol should be introduced before the concept represented by that symbol has developed in the child. For example, Cobb (1995), discusses whether the *conventional* mathematical symbol drives conceptual development (a sociocultural viewpoint) or whether the construction of concepts precedes symbolization (a constructivist perspective). These questions are given a new slant in this investigation, where a *non-conventional,* symbol entailling no cultural baggage, is constructed in order to encapsulate a novel concept (a translocation along a number sequence). In other words, a socio-cultural *form* of tool (the sign), is adapted to serve a specific and relatively individual purpose.

Perhaps the present study - or further studies of this kind might serve as a proving ground for a comparison of many of the specific claims made by both socioculturists and constructivists and provide a partial solution to the question asked by Kaput (1991) *'How do material notations and mental constructions interact to produce new constructions?'* (quoted by Cobb, 1995)

Theoretical·assumptions

Children's 'unary' view of addition has been been known and investigated for some time (eg Baroody & Ginsburg, 1987). Martin Hughes (1986) has also shown that children do not interpret the '+' sign as representing a 'symmetrical' relation between two sets, but more as a sign for action to be taken - to make a number *more* by adding to it another number, or in practical terms, a belief in the noncommutativity of addition. On the basis of this finding, it was proposed to set up a learning subculture where this 'transformational' intuition could be exploited rather than discouraged. For example, by providing a communicative context in which it was necessary for children to invent their own symbols for translocations along a 'concretised' number sequence, it might be possible to develop more effective mental counting strategies. The experimental design assumed the following:-

1. The sequence of counting numbers (1,2,3, ...) is the basis for early addition and subtraction which together with counting techniques is sufficient to solve simple number problems (Fuson, 1988).
2. Counting techniques entail similar mental strategies to those involved in 'transformational' addition (ie. addition in which one number assumes the role of operator and the other the role of operand).
3. Children's problems with early mathematical work in school may be due in great part to the unrelatedness of conventional operation signs to the mental operations which children actually use (Womack, 1992, 1995).

Experimental methodology and the role of the researcher

Since children's mathematical notions *can* be significantly influenced by the nature of the activities they carry out in the broad school setting (Cobb & Yang, 1995), an attempt was made to find a setting in which transformational addition would essential. A 'stepping stone' scenario was finally decided upon since this would relate well to the child's world of action and movement, rather than the 'static' relation of one collection of objects to another. In this 'model', children's transformational intuitions are exploited in the posing of questions about *actions* and *positions,* rather than about *objects* and *sets.*

Vygotsky stresses the profound importance of sociocultural factors, particularly the cultural tools we know as symbols. In the case ofteaching addition and subtraction, the symbols '+' and '-' have enormous cultural influence and for this reason no mention of conventional mathematical terminology was made in the sessions. In fact it was 8 or 10 weeks before the children saw any connection between what they were doing and the written additions and subtractions they were performing in their normal school class.

Vygotsky also showed that the psychological functions on which written speech is based have not begun when instruction in arithmetic starts - but that,

'the development of the psychological foundations for instruction in basic subjects does not precede instruction but unfolds in a continuous interaction with the contributions of instruction. '

(Vygotsky, 1962, plOt).

This 'continuous interaction' of adult instructor with children was to be the basis for the underlying 'pedagogy' (such as it was) within the group. In each session the researcher initiated various scenarios from which questions about numbers arose and in which various types of instruction were necessary. The *need* for written symbolism was also instigated by the researcher. Questioning was generally carried out very informally and responses were tape-recorded and written down immediately after the session (cf. method of Carraher, Carraher & Schliemann (1985) in the context of Brazilian children in Recife). In no circumstance was pressure brought on any child to produce a 'correct' answer. Questions which were put explicitly to children were often asked to pairs of children or a 'team' and the researcher was often a participant in the 'game' or number-guessing scenario.

Needs were created for children to solve simple number puzzles based on sequential actions or 'transformational' scenarios. The need for children to communicate these puzzle situations amongst themselves was created using both oral and written instructions. The presentational format of the written instructions was guided towards a format which used arrows but was similar to a conventional number sentence.These 'transformational number sentences' were then used to pose questions which required mental counting strategies of the same kind used to solve more traditional number questions. The structure of activities for the next session was also determined during the preceeding session. The ultimate aim was to move towards a pedagogy whereby the children's intuitive system could be eventually' grafted' onto the more generalised and abstract, conventional system of arithmetic.

Results

In discussing children's intuitive multiplication models, Mulligan (1997) cites research evidence to suggest that children's use of intuitive models are more dependent on appropriate calculation strategies than the semantic structure of the problems encountered. In the context of this investigation, where the 'model' very closely resembles the semantic structure of the 'problem' situations, this seems a likely hypothesis.

The children successfully invented and used arrows to indicate 'movement' up and down the number sequence and also to determine the number of steps between any two numbers of the number sequence. The way in which these developed have interesting parallels with the notation-types found by Sinclair et al (1983).

Children also showed a good deal of ability in working out number 'problems' using some form of intemalisation of the practical number sequence models (such as stepping stones) they had been using. The 6 year old children were also able to mentally supply missing numbers in transformational 'equations' - which used arrows but no 'equals' sign. For example, in the action symbolized as *A transformed by B results in* C, they could find either A, B or C when the other two terms were given. The 6 year olds were generally successful in this but the 5 year oIds were less so - especially when the transformation was 'negative' (as defined by Vergnaud, 1982), and involved a movement *down* the number sequence.

The problems solved by the children were equivalent to algebraic problems of the form:

$y + A = B$ $A + y = B$ $A - Y = B$ $Y - A = B$ (y is unknown, A, B are known)

Implications and Conclusion

An underlying belief sustaining the programme has been that children's intuitions might form the basis for a consistent and valid 'intuitive mathematics' (or 'transformational arithmetic') which may yield new insights and perspectives on traditional mathematical concepts? We can only speculate what sort of principles might hold in such a world of *'proto-mathematics'*. For example, would they explain the reasoning behind incidents such as the following, which took place during one of the sessions?

I asked one 5 year old (TARA), how many letters were in her name. She replied 'Three' . Later, I counted out aloud with her, the letters in her name. On reaching the second A, she interjected and said 'You've said that one' !

Although we are not yet in a position to elaborate, it seems we may soon find general principles of meta-semiosis which might explain such reactions of young children to symbols.

The research continues.

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