COPING WITH THE REQUIREMENTS FOR RIGOUR: THE NOVELTY OF UNIVERSITY MATHEMATICS

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Abstract

Given that the first-year mathematics undergraduates enter the university with A -level experiences that do not include any formal introduction to proving, the encounter with their course's requirements for rigour is deeply problematic. Based on the findings from a doctoral research project on the novice mathematician's learning difficulties in the encounter with mathematical abstraction, this paper samples a number of these difficulties: extracts of conversations between Oxford tutors and their students are presented and discussed from the point of view of highlighting the difficulty of switching from informal (school) to formal (university) mathematical thinking.

Part I Background and Methodology of a Study of Learning Difficulties at University Level

A psychological study oflearning difficulties in mathematics is intrinsically interesting both to psychologists - as a context-specific application and extension ofpsychological theories - and to mathematics educators as a theoretical contribution to the discourse relating to their pedagogical practices (Tall 1991b). In this sense the doctoral study on which this paper is based (Nardi 1996) is an attempt to contribute, in an implicit and theoretical way, to the pedagogical discourse on the teaching of advanced mathematics.

The transition to abstract mathematical practices both in the sense of adopting axiomatic deduction and of being involved with objects that are not necessarily of physical or numerical nature is considered to be a major difficulty in learning advanced mathematics (as suggested in the literature on Advanced Mathematical Thinking, e.g. (Tall 1991a». In (Nardi *1996)formalisation* - the adoption offormal mathematical reasoning - and *concept-image construction* - the acquisition of new concepts - are the two axes around which the discourse on the difficulties oflearning advanced mathematics is developed.

With regard to exploring the difficulties of the transition to mathematical abstraction, the first year of undergraduate mathematical studies is a crucial turning point: it constitutes the period of transition from the concreteness of school-mathematical <u>thinking</u> to the abstraction of advanced mathematical <u>thinking</u>. (Nardi 1996) aimed at an analysis of the difficulties of this transition.

The study was carried out in Oxford where mathematics undergraduates, in pairs or individually, are offered weekly and approximately hourly tutorials. This is a uniquely intimate learning environment in which students and tutors discuss the contents of problem sheets given to the students in advance and on which the students have worked for a few days before the tutorial. The problem sheets cover a wide range of topics. For the purposes of this study the collected and analysed material was related to Foundational Analysis, Calculus, Linear Algebra, Topology and Group Theory.

The study was qualitative and aimed at the generation of *data-grounded theory* (Glaser and Strauss 1967). Espousing a *phenomenological* approach (Burrell and Morgan 1978) - the phenomena here being the novice mathematician's articulations of their thin kin g - it aimed at the construction of a psychological profile of the difficulties of the induction to mathematical abstraction.

The methodological techniques used were *Minimally-Participant Observation* (of Oxford Tutorials to 20 first-year mathematics undergraduates lasting 14 weeks over 2 terms and 200 hours of tutorial attendance) and *Semi-Structured Interviews* (two thirty-minute ones with each student at the end of each term on issues emerging during observation). The observed tutorials and the interviews were audiorecorded. Eventually a hundred and fifty hours of recorded material was collected.

Selective Transcription of the Audio-Recorded Material followed and significant *Episodes* (stories on learning which illustrate the difficulties of the novice's enculturation into advanced mathematical thinking) were then *extracted*. For each Episode an *Analytical Text* was constructed consisting of a Selective Transcript and the Mathematical Background of the Episode, its Analysis· and a Conclusion representing the essence of the Analysis. The Analytical Texts were ordered in terms of their mathematical content into the five topical areas mentioned above and a *Topical Synthesis* was constructed within each mathematical area. The overall findings were then presented in a *Cross-Topical Synthesis* of the psychological phenomena emerging in the Topical Syntheses.

Drawing both on influences from the *Piagetian* as well as *sociocultural Psychology*, the Cross-Topical Synthesis can be described as an account of the Novice Mathematician's Encounter with Mathematical Abstraction both as *a personal meaning-construction process* and as *an enculturation process* - where the new culture that the novice is introduced to is University Mathematics. Both perspectives are applied on all of the themes emerging from the Data Analysis. The findings are presented in two groups relating to the novices' difficulties with *concept-image construction andformalisation.*

• Major influences and general directives for the Analysis came from *Discourse Analysis, Cognitive and Cultural Psychology* whereas crucial tools for the Analysis were the Piagetian notion of *Reflective Abstraction* and works within Advanced Mathematical Thinking mostly the *Theory of Epistemological Obstacles*, the *Theory of Concept Image Concept Definition* and the *Theory ofReification and Metaphor Construction.* For a comprehensive list of references see (Nardi 1996).

For the purposes of this paper a sample of the themes on *Formalisation* is presented. Formalisation here is used both as *adoption* of the *semantics* offormal mathematics as well as of formal mathematical *reasoning*. In the next section I discuss one aspect of the novices' difficulties in adopting formal mathematical reasoning.

Part II A Theme of Findings: The Novices' Adoption of Formal Mathematical Practices Hampered By Their Ambiguous Perception of What Knowledge Can Be Assumed

As discussed briefly in the previous part the novices in this study were found to be in *difficulty with the mechanics of formal mathematical reasoning.* In this part I exemplify one aspect of these difficulties mostly within the specific context of Foundational Analysis and Calculus: the novices' obselVed *deficiency in embedding the newly-acquired tools offormal mathematical reasoning.* The analysis of these obselVations attributed this deficiency partly to the students' *ambiguity with regard to what knowledge they are allowed to assume.*

In a number of occasions - mostly in the context of proving statements within Foundational Analysis the students' proofs turned out inefficient not only because of their limited proving skills but because the students were not at ease with the assumptions they were allowed to make. Part of the problem seemed to be caused by the absence of clarifications about, for instance, what statements regarding the real numbers can be assumed. The students seem to be extremely vulnerable to such an absence and as a result *over- and under- react to the requirements for rigour.*

In one case a student set out to prove Completeness because he had not realised that Completeness is assumed. Similarly another student, in a state of uncertainty about assuming previously proved statements, deliberately avoided using Bernoulli's inequality in a proof and preferred the more *'basic'* and (for him) established formula of the Binomial Theorem In contrast, the same student who had not realised that Completeness is assumed, in one of his written drafts discussed in the tutorials, proposed the use of Mathematical Induction but did not carry out the proof as he himself explained to the tutor, he thought that since Mathematical Induction was introduced in the course the week before, now *'it could be assumed'* and used by mere reference. In a similar vein another student proposed proving the implication from $P(n)$ to $P(n+1)$ - the third step of Mathematical Induction - as a *'quicker way'* than the complete process of Mathematical Induction. In these cases, the students seem to have been sensitised to the increased requirements of rigour in the new course but then abandoned to clarify these requirements on their own. At the same time they do not seem to have acquired adequate understanding of the mechanics of Mathematical Induction. Their *vulnerability results in inefficiency or avoidance of rigour.*

A basic aspect of their perplexity is *to what degree assuming knowledge is compatible with the requirements for axiomatic rigour* made by the lecturers and tutors in the beginning of the course. In this sense the *tension between Proof-By-First-Principles and Proof-By- Theorem-Quoting* is a graphic illustration of their ambiguity. This tension was mostly obselVed in the context of limits in Calculus, where the students were undecided as to whether they should find limits via the formal definition (First Principles) or via the algebra of limits (Theorem Quoting): using inequalities in order to manipulate quantities, graphing functions, guessing limits and using the algebra of limits are mathematical practices questioned by the novices as to their rigour and, hence, as to their acceptability. Due to their *growing mistrust towards the practices of school mathematics* - against the use of which they are repeatedly advised - they avoid intuitive practices such as guessing limits and then proving them Moreover they are not at ease with the alternation of practices (the use of the definition of limit alternating with the use of the algebra of limits). They also seem weak in distinguishing between the practices that they are supposed to espouse in different mathematical domains: in applied mathematics, retrospective use of unproved results is allowed, as opposed to foundational courses, where it is not. The students in the first term of their studies were taught both an applied and a foundational course and the varying expectations of mathematical rigour in the two courses appear to be contradicting each other.

Mistrust towards intuition does not lead to total elimination of practices such as *reliance on pictorial evidence or intuitive belief* To prove that lim(11=O, when |~ < 1, a student used in his proof his belief that the powers of () go to zero when () is a number less than 1. In another case two students were reluctant to engage in a prooffor the Triangle Inequality for n numbers because they had been convinced by the picture used by the lecturer for the case $n = 2$. So, except for the difficulty with accepting the necessity of proof: embracing rigorous mathematical behaviour is hampered by *vagueness about the rules of the game.* A result of the students' ambiguity about the legitimacy of certain practices is the fragility of their knowledge and, hence, the inefficiency of their action.

One necessary aspect of the novices' enculturation into formal mathematical reasoning is the *reconciliation* between *intuitive mathematical practices* as a way to gain mathematical insight and *formal mathematical language* as a way to refine and establish these insights rigorously. The novices' perplexity about the status of rigour the various approaches carry seems to result in a misunderstanding: the novices are advised to leave behind their school-mathematical way ofthlnking and start anew by trying to build mathematics on the solid foundations of mathematical formalism The novices interpret this suggestion in an exaggerated literal manner and turn suspicious about intuitive mathematical practices. As a result they are cognitively tom between what they instinctively know as a powerful way into mathematical insight (intuition) and their desire to be accepted in the culture of mathematical formalism The *expert's enculturating role then is elevated from the strictly mathematically-topical to the meta-topical,* to demystifying not only particular proofs and solutions but also the rules of the game. I am closing this concise reference to this aspect of the novices' difficulties in their encounter with mathematical rigour with an extract of a conversation between two students and their tutor which illustrates the necessity of the tutor's meta-topical enculturating role. Conversations such as the one illustrated below were not typical at all in the observed tutorials - despite the evidence of the students' need for them -but, in its rarity, the dialectics in this extract may indicate a way towards the kind of meta-mathematical discourse needed in introductory mathematics courses. Moreover the imminent return of the notion of formal proof in the curriculum of school mathematics implies that the preoccupations expressed below by Andrew and Jack can soon be expected to appear in the mathematics classroom too.

The extract is the concluding part of the conversation between a tutor and two students. (T) is the tutor and (A) and (J) are students Andrew and Jack. This is the third week into Andrews and Jack first year and their tutorial has been about proving statements in Foundational Analysis:

A: *Well, if you do something like that would you have to say right in the beginning. Fine, I'll prove this by induction, then by this inequality that was question...*

T: *Well, if you've done it before can't you refer back to it? We always go back to theorems and...*

A: *Yes, it depends... but in the exams you don't know that we've done it.*

T: *I know you've done it! As long as you're clear. You see, your job is to produce proofs which you know really are proofs and which are clear to the person you are trying to communicate with. That he can understand these proofs. That's what you have to do.*

J: *When we came here they said that we have to wipe out all knowledge of math, out of our mind... and now we start assuming things that we learned since we've been here.*

T: *That's the idea, yes. Two weeks ago you knew absolutely nothing. Last week you proved by induction this inequality. So this week...*

A: *So we can have the data... all kinds of uses...*

Part III Cu"ent Plans and Potential Future Directions

As mentioned in Part I, (Nardi 1996) aimed at contributing theoretically to the understanding of the difficulties of advanced mathematical cognition and, thus, implicitly to the development of a pedagogical discourse on the teaching of advanced mathematics. As a first stage to embeddiing the findings of this study in this pedagogical discourse, a follow-up study is now in the process of development. fu this, university mathematics teachers will be presented with the findings of the study - 'translated' for this purpose to a language which is accessible to an audience not necessarily familiar with the language of Psychology of Mathematics Education - and asked to discuss issues emerging from the data analysis. Discussions with the university mathematics teachers are planned as a series of semi- structured interviews in which the interviewee will be presented with a piece of data as well as the analysis of this piece in (Nardi 1996). They will then be asked to comment on the piece. Reflection of and on their own practices is the main aim of these interviews.

References

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